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VERMA TYPE MODULES OF LEVEL ZERO FOR AFFINE LIE ALGEBRAS

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Abstract. We study the structure of Verma type modules of level zero induced from non-standard Borel subalgebras of an affine Kac-Moody algebra. For such modules in "general position" we describe the unique irreducible quotients, construct a BGG type resolution and prove the BGG duality in certain categories. All results are extended to generalized Verma type modules of zero level.

Introduction

One of the significant differences between finite-dimensional and affine Kac-Moody algebras is the existence of root system partitions which are not "equivalent" to the standard partition into positive and negative roots. All $W \times \{\pm 1\}$-inequivalent partitions for affine root systems (W is the Weyl group) were classified in [10], [11] and in [5], [6]. There exist only a finite number of them (always more than one) and each such partition, labeled by some finite set of integers X, defines a non-standard Borel subalgebra B_X of the affine Lie algebra $\mathfrak{G}$ and a Verma type module $M_X(\lambda)$ induced from B_X. Verma type modules were introduced in [10] and [6]. The main difference between the classical Verma modules [12] and the modules induced from the non-standard Borel subalgebras is that the latter always have both finite and infinite-dimensional weight spaces. Verma type modules of a non-zero level, i.e. when the central element acts with a non-zero charge, were extensively studied in [1] and [9]. In this case the structure of a module $M_X(\lambda)$ is completely determined by its subspace $M^f(\lambda)$ containing all finite-dimensional weight subspaces of $M_X(\lambda)$. The subspace $M^f(\lambda)$ has a module structure for a certain infinite-dimensional Lie subalgebra $\tilde{\mathfrak{G}}^f$ with a triangular decomposition [13], and, when the central element acts with a non-zero charge, any submodule $N \subset M_X(\lambda)$ can be recovered from $N^f = N \cap M_X^f(\lambda)$. This leads to the equivalence between a certain category $\mathfrak{O}_\lambda^X$ of $\mathfrak{G}$-modules and a certain category of $\tilde{\mathfrak{G}}^f$-modules, which implies the BGG duality in $\mathfrak{O}_\lambda^X$ and a BGG type resolution for $M_X(\lambda)$ [2]. These results were extended in [3] for the generalized Verma type modules of a non-zero level induced from a non-standard parabolic subalgebra.

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In the present paper we study the Verma type modules on which the central element acts with a zero charge. When $X = \emptyset$ such modules were considered in [10], [11], [6] and [7]. The case of a zero level is apriori more difficult than that of a non-zero level since the module $M_X(\lambda)$ may have subquotients that are not the quotients of Verma type modules [7]. Nevertheless, a similar approach can be developed for λ in "general position". In particular, we describe the submodules and the irreducible quotients $L_X(\lambda)$ of $M_X(\lambda)$ when λ is in "general position". The equivalence of suitable categories for "general position" then leads to the BGG duality in a certain category $\mathfrak{O}_X(\lambda)$ and to the construction of a strong BGG resolution for $L_X(\lambda_0)$, X connected, λ_0 trivial. All results are extended for generalized Verma modules of zero level induced from a non-standard parabolic subalgebra.

Now we briefly describe the structure of the paper. In section 2 we recall the construction and the basic properties of Verma type modules and generalized Verma type modules. In section 3 we establish an important technical result (Proposition 3.2) and discuss the properties of the imaginary Verma modules $M_\emptyset(\lambda)$.

Since the central element of $\mathfrak{G}$ acts trivially, the module $M_X(\lambda)$ is reducible and can be substituted by a certain quotient $\tilde{M}_X(\lambda)$. The central result of section 4 is Theorem 4.8, which establishes the criterion of irreducibility for modules $\tilde{M}_X(\lambda)$. Section 5 is devoted to the study of modules $M_X(\lambda)$ under the assumption that $\tilde{M}_X(\lambda)$ is irreducible. In section 6 we discuss the irreducible quotients of $M_X(\lambda)$ (Theorem 6.1) in the particular case when $\tilde{M}_X^f(\lambda)$ is an irreducible $\tilde{\mathfrak{G}}^f$-module. We lift that restriction in section 7 and describe the irreducible quotients of $M_X(\lambda)$ for λ in "general position" (Theorem 7.7). A strong BGG resolution for modules $L_X(\lambda_0)$ with connected X and trivial λ_0 is constructed in section 8 (Theorem 8.2), and the BGG duality in certain categories $\mathfrak{O}_X(\lambda)$ of $\mathfrak{G}$-modules with λ in "general position" is established in section 9 (Theorem 9.6). The generalized Verma type modules of level zero are discussed in section 10, and suitable categories $\mathfrak{O}_{X,S}(\lambda, q)$ with the BGG duality in section 11. Some subcategories of $\mathfrak{O}_{X,S}(\lambda, q)$ with the BGG duality are considered in section 12.

1. Preliminaries

Let $\mathbb{C}$ denote the complex numbers, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, $A = (a_{ij})$, $0 \le i, j \le n$, be a generalized Cartan matrix of affine type and $\mathfrak{G} = \mathfrak{G}(A)$ be the corresponding affine Kac-Moody algebra of rank $n+1$ with a Cartan subalgebra $\mathfrak{H}$ and a one-dimensional centre $Z = \mathbb{C}c \subset \mathfrak{H}$. Let also $\Delta = \Delta^{re} \cup \Delta^{im}$ be the root system of $\mathfrak{G}$, where Δ^{re} is the set of real roots, $\Delta^{im} = \{k\delta | k \in \mathbb{Z} \setminus \{0\}\}$ is the set of imaginary roots and δ is an indivisible imaginary root. We use [12] as our main reference for Kac-Moody algebras. It follows from [5] that one can choose a basis $\pi_0 = \{\alpha_0, \alpha_1, \dots, \alpha_n\}$ of Δ such that $\delta = \sum_{i=0}^n k_i \alpha_i$, where $k_0 = 1$ and either $-\alpha_0 + \delta \in \Delta$ or $\frac{1}{2}(-\alpha_0 + \delta) \in \Delta$. Each $\alpha \in \Delta$ defines a root subspace $\mathfrak{G}_\alpha$, and $\mathfrak{G} = \mathfrak{H} \oplus \sum_{\alpha \in \Delta} \mathfrak{G}_\alpha$. Fix a basis X_α in each $\mathfrak{G}_\alpha$, $\alpha \in \Delta$. Unless otherwise stated we will always refer to this fixed basis.

For $\epsilon \subset \pi_0$ let $Q_\epsilon^\pm$ be the semigroup in $\mathfrak{H}^*$ generated by $\pm\epsilon$, Q_ϵ be the free abelian group generated by ϵ, $\Delta(\epsilon) = Q_\epsilon \cap \Delta$, $\Delta_\pm(\epsilon) = Q_\epsilon^\pm \cap \Delta$. Set $\pi = \pi_0 \setminus \{\alpha_0\}$, $Q = Q_{\pi_0}$, $\dot{Q} = Q_\pi$, $\dot{\Delta} = \Delta(\pi)$, $\dot{\Delta}_\pm = \Delta_\pm(\pi)$. A subset $P \subset \Delta$ is called a *partition* if P is closed under addition (i.e. α, $\beta \in P$, $\alpha + \beta \in \Delta$ imply $\alpha + \beta \in P$), $P \cap -P = \emptyset$ and $P \cup -P = \Delta$.

Let $I = \{1, 2, \dots, n\}$, $X \subset I$, $\phi_X = \sum_{i \in I \setminus X} \alpha_i^* - (\sum_{i \in I \setminus X} k_i)\alpha_0^*$ if $X \neq I$ and $\phi_I = \sum_{i=0}^n \alpha_i^*$, where $\alpha_i^*(\alpha_j) = \delta_{ij}$, $i, j = 0, 1, \dots, n$. Define $P(X) = \{\alpha \in$

$\Delta|\phi_X(\alpha)>0\}\cup\{\alpha\in\Delta|\phi_X(\alpha)=0,\ \phi_I(\alpha)>0\}$. It was shown in [10] and [5] that any partition P is $W\times\{\pm 1\}$-equivalent to some $P(X)$, where W is the Weyl group of Δ.

We will fix X throughout the paper. Note that if $X=I$ then $P(X)=\Delta_+(\pi_0)$.

Let $(\cdot,\cdot)$ be the standard form on $\mathfrak{H}^*$ such that

$$\frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}=a_{ij},\ i,\ j=0,\ \dots,\ n,$$

$\pi^f=\{\alpha_i, i\in X\}$, $\pi'=\pi\setminus\pi^f$, $\tilde{\pi}=\{\alpha\in\pi'\mid(\alpha,\beta)=0 \text{ for all } \beta\in\pi^f\}$, $Q^f=Q_{\pi^f}+\mathbb{Z}\delta$, $\Delta^f=Q^f\cap\Delta$, $\dot{\Delta}^f_\pm=\Delta^f\cap\dot{\Delta}_\pm$, $\Delta^f_\pm=\Delta^f\cap(\pm P(X))$, $\tilde{\Delta}_\pm=\Delta(\tilde{\pi})\cap\dot{\Delta}_\pm$, $\Delta'_\pm=\dot{\Delta}_\pm\setminus\Delta(\pi^f\cup\tilde{\pi})$, and let $Q^f_\pm$ (resp. $Q_\pm$) be the monoid generated by $\Delta^f_\pm$ (resp. $\pm P(X)$).

For a Lie subalgebra $\mathfrak{A}\subset\mathfrak{G}$, $U(\mathfrak{A})$ denotes the universal enveloping algebra of $\mathfrak{A}$. Clearly, $U(\mathfrak{A})$ is a Q-graded algebra,

$$U(\mathfrak{A})=\bigoplus_{\eta\in Q}U(\mathfrak{A})_\eta.$$

An element $u\in U(\mathfrak{A})$ is called *homogeneous* if $u\in U(\mathfrak{A})_\eta$ for some $\eta\in Q$. We will identify $\mathfrak{G}$ with its injective image in $U(\mathfrak{G})$. If $y,y_1,\dots,y_m\in\mathfrak{G}$ we set

$$[y,y_1,\dots,y_m]=\sum_{i=1}^m y_1\dots y_{i-1}[y,y_i]y_{i+1}\dots y_m$$

and then define $[y,u]$ for any $u\in U(\mathfrak{G})$ by linearity.

For $\epsilon\subset\pi$, $\phi=\dot{\phi}+n\delta\in Q$, $\dot{\phi}\in\dot{Q}$, $n\in\mathbb{Z}$ and $0\neq u\in U(\mathfrak{G})_\phi$ denote by $ht_\epsilon(\phi)$ the number of elements of $\pm\epsilon$ in the decomposition of $\dot{\phi}$, and let $ht_\epsilon(u)=ht_\epsilon(\phi)$, $|u|=n$, $\parallel u\parallel=|n|$. We also set $ht^f(\phi)=ht^f(u)=ht_{\pi^f}(\dot{\phi})$ and $|u|_+=\sum_{i=1}^m|n_i|$ if u is a monomial in $U(\mathfrak{G})$, $u=X_{\phi_1+n_1\delta}\dots X_{\phi_m+n_m\delta},\ \phi_i\in\dot{Q}$.

Define $\mathfrak{G}^\pm_X=\sum\mathfrak{G}_{\pm\beta},\ \beta\in P(X)$. Then we have an X-analog of the Cartan decomposition $\mathfrak{G}=\mathfrak{G}^-_X\oplus\mathfrak{H}\oplus\mathfrak{G}^+_X$. A subalgebra $B_X=\mathfrak{H}\oplus\mathfrak{G}^+_X$ is called a *non-standard Borel subalgebra*.

For $\epsilon\subset\pi'$ consider the subalgebras $\mathfrak{G}_\pm(\epsilon)=\sum\mathfrak{G}_\beta,\ \beta\in(Q^\pm_\epsilon+\mathbb{Z}\delta)\cap\Delta^{re}$, and let $\mathfrak{G}(\epsilon)$ be a subalgebra of $\mathfrak{G}$ generated by $\mathfrak{G}_\pm(\epsilon)$. In particular, set $\tilde{\mathfrak{G}}_\pm=\mathfrak{G}_\pm(\tilde{\pi})$, $\tilde{\mathfrak{G}}=\mathfrak{G}(\tilde{\pi})$. Also let $\mathfrak{G}'_\pm=\sum\mathfrak{G}_{\pm\beta},\ \beta\in P(X),\ ht'(\beta)\neq 0$, $\mathfrak{G}^f_\pm$ (resp. $\mathfrak{G}^f$) be a subalgebra generated by $\Delta^f_\pm\cap\Delta^{re}$ (resp. $\Delta^f\cap\Delta^{re}$), $\mathbf{m}^\pm=\sum\mathfrak{G}_\beta,\ \beta\in\Delta^f_\pm$, $\mathbf{m}=\mathbf{m}^-\oplus\mathfrak{H}\oplus\mathbf{m}^+$ and $\tilde{\mathfrak{G}}^f=\mathfrak{G}^f+\mathfrak{H}$. Clearly, $[\tilde{\mathfrak{G}},\mathfrak{G}^f]=0$ and $\mathfrak{G}^\pm_X=\mathbf{m}^\pm\oplus\mathfrak{G}'_\pm\oplus\tilde{\mathfrak{G}}_\pm$. If the Coxeter-Dynkin subdiagram corresponding to X is connected, then $\mathfrak{G}^f$ is the derived algebra of an affine Lie algebra of rank $|X|+1$ with a root system Δ^f. If X is not connected then Δ^f does not have a basis consisting of real roots. Nevertheless, $\tilde{\mathfrak{G}}^f$ (resp. $\mathbf{m}$) is a Lie algebra with a triangular decomposition [13] with respect to Q (cf. [2], Remark 1.4) and it satisfies the conditions (T1) and (T2) of [14]. Let $X=\bigcup_{i=1}^m X_i$, and let the diagrams corresponding to each X_i be connected. Then

$$\mathfrak{G}^f=\sum_{i=1}^m\mathfrak{G}^f_i,\ [\mathfrak{G}^f_i,\mathfrak{G}^f_j]=0, i\neq j,\ \bigcap_{i=1}^m\mathfrak{G}^f_i=Z$$

and $\mathfrak{G}^f_i$ is the derived algebra of an affine Lie algebra of rank $|X_i|+1$ for each i.

Let

$$G_\pm=\sum\mathfrak{G}_{\pm k\delta},\ k\in\mathbb{Z}_+\setminus\{0\},\ G=G_-\oplus Z\oplus G_+,$$

$$\bar{G} = \{g \in G_- \oplus G_+ \mid [g, \mathfrak{G}^f] = 0\},\ \bar{G}_\pm = \bar{G} \cap G_\pm.$$

Then $G = (G \cap \mathfrak{G}^f) \oplus \bar{G}$ (cf. [6]), $\mathbf{m} = \tilde{\mathfrak{G}}^f \oplus \bar{G}$, $\mathbf{m}^\pm = \mathfrak{G}^f_\pm \oplus \bar{G}_\pm$. Also consider the subalgebras $\mathbf{u}_X^\pm = \sum \mathfrak{G}_{\pm\beta}$, $\beta \in P(X) \setminus \Delta^f$, and $\mathbf{p}_X = \mathbf{u}_X^+ \oplus \mathbf{m}$.

Let $\mathfrak{A}$ be a Lie subalgebra of $\mathfrak{G}$ and $\mathfrak{H} \subset \mathfrak{A}$. An $\mathfrak{A}$–module V is called *weight* if

$$V = \bigoplus_{\lambda \in \mathfrak{H}^*} V_\lambda,\ V_\lambda = \{v \in V \mid hv = \lambda(h)v \text{ for all } h \in \mathfrak{H}\}.$$

Set $P(V) = \{\lambda \in \mathfrak{H}^* \mid V_\lambda \neq 0\}$. For $\lambda,\ \mu \in \mathfrak{H}^*$, we say $\mu \leq \lambda$ if $\lambda - \mu \in Q_+^f$.

Let Ω be a subcategory of weight $\mathfrak{A}$-modules with irreducible objects $\{L_t,\ t \in T\}$ indexed by a certain subset $T \subset \mathfrak{H}^*$. An $\mathfrak{A}$–module $V \in \Omega$ has a local composition series [4] if for any $\mu \in P(V)$ there exist a sequence $V = V_n \supset \ldots \supset V_0 = 0$ of modules in Ω and a subset $J \subset \{1, 2, \ldots, n\}$ such that

1. If $i \in J$, then $V_i/V_{i-1} = L_{t_i},\ t_i - \mu \in Q_+$.
2. If $i \notin J$, then $(V_i/V_{i-1})_\nu = 0$ for all $\nu \in \mu + Q_+$.

We will denote by $[V : L_t]$ the multiplicity of L_t in V, i.e. the number of i's in J such that $t = t_i$.

2. Verma and generalized Verma type modules

Let $\lambda \in \mathfrak{H}^*$, and let $\mathbb{C}v_\lambda$ be the 1-dimensional B_X-module, where $\mathfrak{G}_X^+ v_\lambda = 0$ and $hv_\lambda = \lambda(h)v_\lambda$ for all $h \in \mathfrak{H}$. Consider a $\mathfrak{G}$-module

$$M_X(\lambda) = U(\mathfrak{G}) \otimes_{U(B_X)} \mathbb{C}v_\lambda$$

associated with X and λ. This module is called a *Verma type module of level* $\lambda(c)$ [8], [9]. It is a weight module, and when $X = I$ the module $M_X(\lambda)$ is a usual Verma module [12]. In this case all weight subspaces are finite-dimensional. If $X \subsetneq I$, then $M_X(\lambda)$ possesses both finite and infinite-dimensional weight spaces, $\mu \in P(M_X(\lambda))$ if and only if $\lambda - \mu \in Q_+$, $0 < \dim M_X(\lambda)_\mu < \infty$ if and only if $\mu \leq \lambda$. It has a unique maximal submodule, and we will denote by $L_X(\lambda)$ the unique irreducible quotient [6]. It follows from the construction that $M_X(\lambda)$ is a free $U(\mathfrak{G}_X^-)$-module.

From now on we will assume that $X \neq I$. Set $M^f(\lambda) = \sum_{\mu \leq \lambda} M_X(\lambda)_\mu$ and $L^f(\lambda) = \sum_{\mu \leq \lambda} L_X(\lambda)_\mu$. Both $M^f(\lambda)$ and $L^f(\lambda)$ are $\mathbf{m}$-modules, and $L^f(\lambda)$ is the unique irreducible quotient of $M^f(\lambda)$. It follows from the construction of $M_X(\lambda)$ that $M^f(\lambda)$ is the Verma $\mathbf{m}$-module with highest weight λ with respect to the triangular decomposition $\mathbf{m} = \mathbf{m}^- \oplus \mathfrak{H} \oplus \mathbf{m}^+$ [13], and in particular it is $\mathbf{m}^-$ –free. We can also view the modules $M^f(\lambda)$ and $L^f(\lambda)$ as $\mathbf{p}_X$-modules with the trivial action of $\mathbf{u}_X^+$.

When $\lambda(c) \neq 0$, the structure of $M_X(\lambda)$ is completely determined by $M^f(\lambda)$, and the irreducible quotients of Verma type modules in this case were described in [1] and [9].

Theorem 2.1 ([1], [9])**.** *Let* $\lambda \in \mathfrak{H}^*$, $\lambda(c) \neq 0$. *Then*

$$L_X(\lambda) \simeq U(\mathfrak{G}) \otimes_{U(\mathbf{p}_X)} L^f(\lambda)$$

Suppose that $X \neq \emptyset$ and let $S \subsetneq \pi^f$, $N_S^\pm = \sum \mathfrak{G}_{\pm\alpha}$, $\alpha \in Q_S^+$, $\mathbf{m}_S = N_S^- \oplus \mathfrak{H} \oplus N_S^+$, $\mathbf{u}_{X,S}^\pm = \sum(\mathfrak{G}_{\pm\alpha} \cap \mathfrak{G}^f)$, $\alpha \in \Delta_+^f \setminus Q_S^+$, $\mathbf{p}_{X,S} = \mathbf{m}_S \oplus \mathbf{u}_{X,S}^+$, $\mathfrak{H}_S = [N_S^-, N_S^+]$, $T_{X,S}^\pm = \mathbf{u}_{X,S}^\pm \oplus \mathbf{u}_X^\pm \oplus \bar{G}_\pm$ and $N_{X,S} = \mathbf{m}_S \oplus T_{X,S}^+ = \mathbf{p}_{X,S} \oplus \mathbf{u}_X^+ \oplus \bar{G}_+$. Thus $\mathbf{m}_S$ is a finite-dimensional reductive Lie algebra, $\tilde{\mathfrak{G}}^f = \mathbf{p}_{X,S} \oplus \mathbf{u}_{X,S}^-$ and $\mathfrak{G}_X^+ =$

$\mathbf{u}_X^+ \oplus \mathbf{u}_{X,S}^+ \oplus N_S^+ \oplus \bar{G}_+$. The subalgebra $N_{X,S}$ is called a *non-standard parabolic subalgebra* [3].

Let $\lambda \in \mathfrak{H}^*$ and λ_S be the restriction of λ to $\mathfrak{H}_S$. Suppose that λ_S is dominant integral and consider a finite-dimensional irreducible $\mathbf{m}_S$-module $V_S(\lambda)$ with highest weight λ (i.e. $\lambda+\alpha \notin P(V_S(\mu))$ for any $\alpha \in S$). We can view $V_S(\lambda)$ as $N_{X,S}$-module with a trivial action of $T_{X,S}^+$ and define the $\mathfrak{G}$-module

$$M_{X,S}(\lambda) = U(\mathfrak{G}) \otimes_{U(N_{X,S})} V_S(\lambda)$$

associated with X, S and λ. The module $M_{X,S}(\lambda)$ is called a *generalized Verma type module*. If $S = \emptyset$ then $M_{X,\emptyset}(\lambda)$ is the Verma type module associated with X and λ. The properties of modules $M_{X,S}(\lambda)$ were discussed in [3]. Clearly, $M_{X,S}(\lambda)$ is a weight module, $M_{X,S}(\lambda) \simeq U(T_{X,S}^-) \otimes_{\mathbb{C}} V_S(\lambda)$ and $L_X(\lambda)$ is the unique irreducible quotient of $M_{X,S}(\lambda)$. Also note that $0 < \dim M_{X,S}(\lambda)_\mu < \infty$ if and only if $\mu \in P(M_{X,S}(\lambda))$ and $\mu \leq \lambda$. Consider the subspace $M_{X,S}^f(\lambda) = \sum M_{X,S}(\lambda)_\mu$, $\dim M_{X,S}(\lambda)_\mu < \infty$, which is an $\mathbf{m}$-module. We can also view $M_{X,S}^f(\lambda)$ as a $\mathbf{p}_X$-module with trivial action of $\mathbf{u}_X^+$.

Since $V_S(\lambda)$ is an $N_{X,S}$-module, it is a $\mathbf{p}_{X,S} \oplus \bar{G}_+$-module where $\mathbf{u}_{X,S}^+$ and $\bar{G}_+$ act trivially.

Proposition 2.2 ([3]). 1. $M_{X,S}^f(\lambda) \simeq U(\mathbf{m}) \otimes_{U(\mathbf{p}_{X,S} \oplus \bar{G}_+)} V_S(\lambda)$.

2. *$L^f(\lambda)$ is the unique irreducible quotient of $M_{X,S}^f(\lambda)$.*
3. *If $\lambda(c) \neq 0$, λ_S is dominant integral and $N \subset M_{X,S}(\lambda)$ is a submodule, then $N \simeq U(\mathfrak{G}) \otimes_U (\mathbf{p}_X) N^f$, where $N^f = N \cap M_{X,S}^f(\lambda)$, and in particular, $M_{X,S}(\lambda) \simeq U(\mathfrak{G}) \otimes_{U(\mathbf{p}_X)} M_{X,S}^f(\lambda)$.*

3. Imaginary Verma modules

In this section we discuss the properties of the modules $M_\emptyset(\lambda)$. Such modules were studied originally in [10], [11] and [7]. Following [7] we call them *imaginary Verma modules*.

First we establish the following technical lemma and proposition which will be used later.

Lemma 3.1. 1. *If $\psi \in \Delta'_+$, $ht'(\psi) = 1$ and $ht^f(\psi) \neq 0$, then there exists $\phi \in \Delta'_+$ such that $\phi - \psi \in \dot{\Delta}_-^f$.*

2. *If $\psi \in \Delta'_+$ (resp. $\psi \in \tilde{\Delta}_+$) and $ht_\pi(\psi) \neq 1$, then there exists $\phi \in \dot{\Delta}_+$ such that $ht'(\phi) = 1$ (resp. $\tilde{ht}(\phi) = 1$) and $\phi - \psi \in \dot{\Delta}_-$.*
3. *Let $\phi \in \Delta'_+$, $ht^f(\phi) = 0$, $n \in \mathbb{Z}$ and $-\phi + n\delta \in \Delta$. Then there exists $m \in \mathbb{Z}$, $m + n \neq 0$, such that $\phi + m\delta \in \Delta$ and*
$$T_{m,n} = [\mathfrak{G}_{\phi+m\delta}, \mathfrak{G}_{-\phi+n\delta}] \not\subset \bar{G}.$$
4. *Let $\phi, \psi \in \dot{\Delta}_+$, $\phi \neq \psi$, $n \in \mathbb{Z}$, $-\psi + n\delta \in \Delta$ and $\phi - \psi \in \dot{\Delta}_-$. Then there exists $m \in \mathbb{Z}_+$ such that $\phi \pm m\delta \in \Delta$ and $\phi - \psi + (n \pm m)\delta \in \Delta$.*
5. *If $\epsilon \subset \pi'$, $\epsilon' \subsetneq \epsilon$, $\epsilon'' \subset \epsilon'$, $\psi \in \Delta_-(\epsilon) \setminus \Delta_-(\epsilon')$ and $ht_{\epsilon''}(\psi) \neq 0$, then there exists $\phi \in \dot{\Delta}_+$ with $ht_{\epsilon''}(\phi) = 1$ and $\phi + \psi \in \Delta_-(\epsilon) \setminus \Delta_-(\epsilon')$.*

Proof. Statements (1), (2) and (5) are obvious. Let $\phi \in \Delta'_+$ and $n \in \mathbb{Z}$ as in (3). Since $\phi \notin \tilde{\Delta}$ and $ht^f(\phi) = 0$, there exists $\beta \in \pi^f$ such that $\beta + \phi \in \Delta'$ and $\beta - \phi \notin \Delta'$. Suppose first that $\mathfrak{G}$ is a non-twisted affine Lie algebra. In this case, for any $m \in \mathbb{Z}$, $m + n \neq 0$, $[\mathfrak{G}_\beta, T_{m,n}] \neq 0$, which implies that $T_{m,n} \not\subset \bar{G}$. Now let $\mathfrak{G}$ be a twisted affine Lie algebra and $\mathfrak{G} \neq D_4^{(3)}$. If n is even then for any even m, $[\mathfrak{G}_\beta, T_{m,n}] \neq 0$ and $T_{m,n} \not\subset \bar{G}$. Assume that n is odd. If $\beta + \delta \in \Delta$ then for any $m \in \mathbb{Z}$, $m + n \neq 0$, $[\mathfrak{G}_{\beta+\delta}, T_{m,n}] \neq 0$, implying that $[\mathfrak{G}_\beta, T_{m,n}] \neq 0$ and $T_{m,n} \not\subset \bar{G}$. Let $\beta + \delta \notin \Delta$. Since $[\mathfrak{G}_\beta, [\mathfrak{G}_{\phi+m\delta}, \mathfrak{G}_\delta]] = \mathfrak{G}_{\beta+\phi+(m+1)\delta} \neq 0$ for any odd m, we conclude that $\beta + \psi + m\delta \in \Delta$ and thus for any odd m, $m + n \neq 0$, $[\mathfrak{G}_\beta, T_{m,n}] \neq 0$ and $T_{m,n} \not\subset \bar{G}$. Suppose now that $\mathfrak{G} = D_4^{(3)}$. If $n \equiv 0(\text{mod } 3)$ then $[\mathfrak{G}_\beta, T_{m,n}] \neq 0$ for any $m \equiv 0(\text{mod } 3)$, $m + n \neq 0$, and hence $T_{m,n} \not\subset \bar{G}$. Let $n \equiv 1(\text{mod } 3)$. Then $\phi + m\delta \in \Delta$ for any $m \equiv 2(\text{mod } 3)$. If $\beta + \delta \in \Delta$ then $[\mathfrak{G}_{\beta+\delta}, T_{m,n}] \neq 0$ for any $m \equiv 2(\text{mod } 3)$. Hence $[\mathfrak{G}_\beta, T_{m,n}] \neq 0$ and $T_{m,n} \not\subset \bar{G}$. Let $\beta + \delta \notin \Delta$. Then $0 \neq [\mathfrak{G}_\beta, [\mathfrak{G}_{\phi+m\delta}, \mathfrak{G}_\delta]] = [\mathfrak{G}_{\beta+\phi+m\delta}, \mathfrak{G}_\delta]$ and $\beta + \phi + m\delta \in \Delta$ for any $m \equiv 2(\text{mod } 3)$. Hence, $[\mathfrak{G}_\beta, T_{m,n}] \neq 0$ and $T_{m,n} \not\subset \bar{G}$ for any $m \equiv 2(\text{mod}3)$. The case when $n \equiv 2(\text{mod } 3)$ can be treated similarly. This completes the proof of (3). The proof of (4) is analogous to the proof of (3). ◻

For $\epsilon \subset \pi'$ and $\mu \in \mathfrak{H}^*$ set $\epsilon(\mu) = \{\beta \in \epsilon \mid (\beta, \mu) = 0\}$, $\mathfrak{G}_\pm(\epsilon, \mu) = \sum \mathfrak{G}_\phi$, $\phi \in (Q_\epsilon^\pm + \mathbb{Z}\delta) \cap \Delta^{re}$, $ht_{\epsilon \setminus \epsilon(\mu)}(\phi) \neq 0$.

Proposition 3.2. *Let V be a weight $\mathfrak{G}$-module, $\mu \in \mathfrak{H}^*$, $\mu(c) = 0$, $0 \neq v_0 \in V_\mu$, $\mathfrak{G}_{n\delta} v_0 = 0$ for all $n > n_0 \geq 0$, $\epsilon \subset \pi'$, $\mathfrak{G}_+(\epsilon) v_0 = 0$, $N = U(\mathfrak{G}_-(\epsilon)) v_0$, and let V be free as $\mathfrak{G}_-(\epsilon, \mu)$-module. Then the following statements are equivalent:*

1. *$v_0 \in U(\mathfrak{G}(\epsilon)) v$ for any non-zero $v \in N$.*
2. *$\beta \in \epsilon(\mu)$ implies $\mathfrak{G}_{-\beta} v_0 = 0$.*

Proof. Let $\tilde{N} = U(\mathfrak{G}(\epsilon)) v$. Suppose that $\beta \in \epsilon(\mu)$ but $\mathfrak{G}_{-\beta} v_0 \neq 0$. Since $\mathfrak{G}_+(\epsilon) v_0 = 0$, $\mathfrak{G}_+(\epsilon) \mathfrak{G}_{-n\delta} v_0 = 0$ for any $n \in \mathbb{Z} \setminus \{0\}$ and $(\mu, \beta) = 0$, we immediately conclude that $U(\mathfrak{G}(\epsilon)) \mathfrak{G}_{-\beta} v_0 \not\ni v_0$ and hence (1) implies (2).

Assume now that $\mathfrak{G}_{-\beta} v_0 = 0$ for all $\beta \in \epsilon(\mu)$. If $\epsilon(\mu) = \epsilon$ then $N \simeq \mathbb{C} v_0$ and the statement (1) is trivial. Let $\epsilon(\mu) \neq \epsilon$ and v an arbitrary non-zero element of N. Then $v = u v_0$, $u \in U(\mathfrak{G}_-(\epsilon))$, and we may assume that u is homogeneous. We divide the proof of statement (1) into several steps.

Step 1. Suppose that $ht_{\epsilon(\mu)}(u) = d > 0$. Using the fact that N is free over $\mathfrak{G}_-(\epsilon, \mu)$ and the PBW Theorem, we can write u as a linear combination of the monomials

$$X_{\phi_{1i}+n_{1i}\delta}^{\ell_{1i}} \dots X_{\phi_{s(i)i}+n_{s(i)i}\delta}^{\ell_{s(i)i}} u_i,$$

where $u_i \in U(\mathfrak{G}_-(\epsilon, \mu))$, $ht_{\epsilon(\mu)}(u_i) = 0$, $ht_{\epsilon \setminus \epsilon(\mu)}(\phi_{ij}) \neq 0$, $\phi_{ij} \in \Delta_-(\epsilon)$ or $\frac{1}{2}\phi_{ij} \in \Delta_-(\epsilon)$ (the latter is possible only if $\mathfrak{G} = A_{2\ell}^{(2)}$), $0 < ht_{\epsilon(\mu)}(\phi_{s(i)i}) \leq \ldots \leq ht_{\epsilon(\mu)}(\phi_{1i})$ for all i and ℓ_{in}, $n_{ij} \in \mathbb{Z}$, $\ell_{ij} > 0$. We can also assume that $ht_{\epsilon(\mu)}(\phi_{s(1)1}) \leq ht_{\epsilon(\mu)}(\phi_{s(i)i})$ for all i, and if $\phi_{s(1)1} + n_{s(1)1}\delta = \phi_{ij} + n_{ij}\delta$ then $i = s(j)$. Suppose that $\phi_{s(1)1} \in \Delta_-(\epsilon)$. Then applying Lemma 3.1, (5) we can find an element $\phi \in \dot{\Delta}_+$ for which $ht_{\epsilon(\mu)}(\phi) = 1$ and $\phi + \phi_{s(1)1} \in \Delta_-(\epsilon)$. Also by Lemma 3.1, (4) one can choose $n \in \mathbb{Z}_+$ such that $n > |u|_+$, $\phi + n\delta \in \Delta$ and $\phi + \phi_{s(1)1} + (n_{s(1)1} + n)\delta \in \Delta$. Since $[y, u_i] = 0$ for $y \in \mathfrak{G}_{\phi+n\delta} \setminus \{0\}$ and for all i, the chosen ordering and n guarantee that $[y, u]$ contains a monomial

$$X_{\phi_{11}+n_{11}\delta}^{\ell_{11}} \dots X_{\phi_{s(1)1}+n_{s(1)1}\delta}^{\ell_{s(1)1}-1} X_{\phi+\phi_{s(1)1}+(n_{s(1)1}+n)\delta} u_1$$

and thus $[y, u] \neq 0$. We conclude that $yv = yuv_0 = [y, u] v_0 \neq 0$ and $ht_{\epsilon(\mu)}([y, u]) = d - 1$. If $\frac{1}{2}\phi_{s(1)1} \in \Delta_-(\epsilon)$ then

$$X_{-\frac{1}{2}\phi_{s(1)1}} v = [X_{-\frac{1}{2}\phi_{s(1)1}}, u] v_0 \neq 0 \text{ and } ht_{\epsilon(\mu)}([X_{-\frac{1}{2}\phi_{s(1)1}}, u]) < d.$$

By induction on d we find an element $z \in \mathfrak{G}_+(\epsilon)$ such that $zv \in N$ and $ht_{\epsilon(\mu)}(zv) = 0$.

We will assume that

$$u = \sum_i c_i X^{\ell_{1i}}_{\phi_{1i}+n_{1i}\delta} \dots X^{\ell_{s(i)i}}_{\phi_{s(i)i}+n_{s(i)i}\delta}, \tag{3.1}$$

where $\sum_{j=1}^{s(i)} \phi_{ji} = \eta \in Q_\epsilon^-$ for all i and $ht_{\epsilon(\mu)}(\phi_{ij}) = 0$ for all i, j.

Step 2. Suppose that $u \in \sum_i c_i X_{\beta_1+n_{1i}\delta} \dots X_{\beta_k+n_{ki}\delta}$, $-\beta_i \in \epsilon \setminus \epsilon(\mu)$, $i = 1, \dots k$, $\beta_1 + \dots + \beta_k = \eta$, $\beta_i \neq \beta_j$ if $i \neq j$. If $n_0 = 0$ then $0 \neq X_{-\beta_1-n_{11}\delta} \dots X_{-\beta_k-n_{k1}\delta} v \in \tilde{N}$ is proportional to v_0 since $\mu([X_{-\beta_i-n_{i1}\delta},\ X_{\beta_i+n_{i1}\delta}]) \neq 0$ for all $i = 1, \dots, k$. Now let $n_0 > 0$. We may also assume that $n_{k1} \leq n_{ki}$ for all i. Denote $y_i = X_{\beta_1+n_{1i}\delta} \dots X_{\beta_{k-1}+n_{k-1i}\delta}$. Then $X_{-\beta_k-n_{k1}\delta} v \in \tilde{N}$ is proportional to

$$w_1 = \sum_{i:n_{ki}=n_{k1}} c_i y_i v_0 + \sum_{i:n_{ki}=n_{k1}+1} c_i' y_i X_\delta v_0 + \dots + \sum_{i:n_{ki}=n_{k1}+p_1} c_i' y_i X_{p_1\delta} v_0,$$

where $p_1 \leq n_0$. Note that

$$-\beta_k + (p_1 - n_{k1})\delta \in \Delta \text{ and } X_{-\beta_k+(p_1-n_{k1})\delta} v \in \tilde{N}$$

is proportional to

$$\begin{aligned} w_2 = &\sum_{i:n_{ki}=n_{k1}} c_i y_i X_{p_1\delta} v_0 \\ &+ \sum_{i:n_{ki}=n_{k1}+1} c_i'' y_i X_{(p_1+1)\delta} v_0 + \dots + \sum_{i:n_{ki}=n_{k1}+p_2} c_i'' y_i X_{(p_1+p_2)\delta} v_0, \end{aligned}$$

where $p_1 + p_2 \leq n_0$. Continuing this process we find $s > 0$ for which $0 \neq w_{s+1} = \sum_{i:n_{ki}=n_{k1}} c_i y_i X_{(p_1+\dots p_s)\delta} v_0 \in \tilde{N}$ and hence $X_{(p_1+\dots+p_s)\delta} v_0 \in \tilde{N}$ by induction on k, implying that $\sum_{i:n_{ki}=n_{k1}+p_s} c_i'' y_i X_{(p_1+\dots+p_s)\delta} v_0 \in \tilde{N}$. Combining the inductions on p_s and s we conclude that $\sum_{i:n_{ki}=n_{k1}} c_i y_i v_0 \in \tilde{N}$, and then induction on k completes the proof.

Step 3. Let $\mathfrak{G} \neq A^{(2)}_{2\ell}$, $u \in U(\mathfrak{G}_-(\epsilon))_{2\beta+m\delta}$, $-\beta \in \epsilon \setminus \epsilon(\mu)$. Then u can be written in the form

$$u = \sum_{k,n} a_{kn} X_{\beta+k\delta} X_{\beta+n\delta}, \tag{3.2}$$

where $a_{kn} \in \mathbb{C}^*$, $k + n = m$, $k \geq n$, and the basis of $\mathfrak{G}$ is chosen in such a way that $[X_{\beta+p\delta},\ X_{q\delta}] = 2X_{\beta+(p+q)\delta}$, $p, q \in \mathbb{Z}$, $q \neq 0$, $[X_{-\beta+p\delta},\ X_{\beta+q\delta}] = X_{(p+q)\delta}$, $p, q \in \mathbb{Z}$, $p + q \neq 0$. We will assume that there are at least two terms in (3.2); otherwise Step 3 can be easily reduced to Step 2. If $p > n_0 - m$ and $-\beta + p\delta \in \Delta$, then $X_{-\beta+p\delta} v = X_{-\beta+p\delta} u v_0 = -2(\sum_{k,n} a_{kn}) X_{\beta+(p+m)\delta} v_0$ and thus we can apply Step 2 if $X_{-\beta+p\delta} v \neq 0$. Suppose that $X_{-\beta+p\delta} v = 0$. Since N is free as $\mathfrak{G}_-(\epsilon, \mu)$-module, we conclude that $\sum_{k,n} a_{kn} = 0$. Let $\bar{n}$ be the minimal number among all n's in a_{kn} and let $n_0 = 0$. Since $(\mu, \beta) \neq 0$, it implies that $X_{-\beta-\bar{n}\delta} v$ is proportional to $X_{\beta+(m-\bar{n})\delta} v_0$ and hence we can apply Step 2. Suppose now that $n_0 > 0$. Since $m - \bar{n} > \bar{n}$, an element of type

$$X_{-\beta-(m-\bar{n})\delta} X_{-\beta-\bar{n}\delta} v$$

is proportional to

$$w_1 = v_0 + b_{11} X_{-n_{11}\delta} X_{n_{11}\delta} v_0 + \dots + b_{k(1)1} X_{-n_{k(1)1}\delta} X_{n_{k(1)1}\delta} v_0 \in \tilde{N},$$

where $b_{i1} \in \mathbb{C}^*$, $0 < n_{i1} < n_{i+11} \leq n_0$ and $X_{n_{i1}\delta}v_0 \neq 0$ for all $i = 1, \ldots, k(1)$. Since $\beta + (n_{k(1)1} + \bar{n})\delta \in \Delta$, it implies that

$$-\beta + (n_{k(1)1} - \bar{n})\delta \in \Delta \text{ and } X_{-\beta-(m-\bar{n})\delta}X_{-\beta+(n_{k(1)1}-\bar{n})\delta}v$$

is proportional to

$$w_2 = X_{n_{k(1)1}\delta}v_0 + b_{12}X_{-n_{12}\delta}X_{n_{12}\delta}v_0 + \ldots + b_{k_2 2}X_{-n_{k(2)2}\delta}X_{n_{k(2)2}\delta}v_0 \in \tilde{N},$$

where $b_{i2} \in \mathbb{C}^*$, $n_{k(1)1} < n_{i2} < n_{i+12} \leq n_0$ and $X_{n_{i2}\delta}v_0 \neq 0$ for all $i = 1, \ldots, k_2$. Continuing this procedure we find $s > 0$ for which

$$0 \neq w_{s+1} = X_{n_{k(s)s}\delta}v_0 \in \tilde{N}, w_s - b_{k(s)s}X_{-n_{k(s)s}\delta}w_{s+1} \in \tilde{N},$$

and we conclude by induction that

$$X_{n_{k(s-1)s-1}\delta}v_0 \in \tilde{N}, \ldots, X_{n_{k(1)1}\delta}v_0 \in \tilde{N},$$

and $v_0 \in \tilde{N}$, which completes the proof.

Step 4. Let $\mathfrak{G} = A_{2\ell}^{(2)}$, $u \in U(\mathfrak{G}_-(\epsilon))_{2\beta+m\delta}$, $-\beta \in \epsilon \setminus \epsilon(\mu)$. Then

$$u = a_m X_{2\beta+m\delta} + \sum_{k,n} a_{kn} X_{\beta+k\delta}X_{\beta+n\delta}, \tag{3.3}$$

where there are at least two terms in (3.3)

$$[X_{\beta+m\delta},\ X_{2k\delta}] = 2X_{\beta+(m+2k)\delta},$$

$$[X_{\beta+m\delta},\ X_{(2k+1)\delta}] = 6X_{\beta+(m+2k+1)\delta},$$

$$[X_{-\beta+m\delta},\ X_{\beta+p\delta}] = X_{(m+p)\delta},$$

$$[X_{-\beta+2m\delta},\ X_{2\beta+(2\ell+1)\delta}] = X_{\beta+(2m+2\ell+1)\delta},$$

$$[X_{-\beta+(2m+1)\delta},\ X_{2\beta+(2\ell+1)\delta}] = -X_{\beta+2(\ell+m+1)\delta},\ p, k, m, \ell \in \mathbb{Z},\ k \neq 0,\ m + p \neq 0;$$

$a_{kn} \in \mathbb{C}^*$, $a_m \in \mathbb{C}$ and $a_m = 0$ if m is even. Let $p \in \mathbb{Z}$ and $2p > n_0 - m$. Then $X_{-\beta+2p\delta}v = (a_m - 2\sum_{k-\text{even}} a_{kn} - 6\sum_{k-\text{odd}} a_{kn})X_{\beta+(m+2p)\delta}v_0$ and

$$X_{-\beta+(2p+1)\delta}v = (-a_m - 6\sum_{k-\text{even}} a_{kn} - 2\sum_{k-\text{odd}} a_{kn})X_{\beta+(m+2p+1)\delta}v_0.$$

If $X_{-\beta+2p\delta}v \neq 0$ or $X_{-\beta+(2p+1)\delta}v \neq 0$ we can apply Step 2; otherwise

$$a_m - 2\sum_{k-\text{even}} a_{kn} - 6\sum_{k-\text{odd}} a_{kn} = a_m + 6\sum_{k-\text{even}} a_{kn} + 2\sum_{k-\text{odd}} a_{kn} = 0$$

and one can show following the procedure in Step 3 that $v_0 \in \tilde{N}$.

Step 5. Suppose that $\eta = 2\beta_1 + \ldots + 2\beta_n$, $-\beta_i \in \epsilon \setminus \epsilon(\mu)$, $i = 1, \ldots, n$, and for each ϕ_{ij} in (1) either $ht_\epsilon(\phi_{ij}) = 1$ or $ht_\epsilon(\phi_{ij}) = 2$ and $\frac{1}{2}\phi_{ij} \in \epsilon$. The proof of statement (1) in this case follows from Steps 3 and 4 using induction on n.

Step 6. Suppose that in (3.1) there exist i and j for which $ht_\epsilon(\phi_{ji}) \geq 2$ and $\frac{1}{2}\phi_{ji} \notin \epsilon$ or there exists $\beta \in -(\epsilon \setminus \epsilon(\mu))$ such that $\eta - 3\beta \in Q_\epsilon^-$. Using Lemma 3.1, (2) and (4) one can find an element $y \in U(\mathfrak{G}_+(\epsilon))$ such that $yv = yuv_0 = u'v_0$ where $u' \in U(\mathfrak{G}_-(\epsilon))$ and it has the same form as in Step 5. We leave the details to the reader. This completes the proof of the proposition. □

Now assume that $X = \emptyset$ and consider the properties of the modules $M_\emptyset(\lambda)$. One can easily see that $0 < \dim M_\emptyset(\lambda)_\mu < \infty$ if and only if $\mu = \lambda - k\delta$, $k \in \mathbb{Z}_+$, which together with Theorem 2.1 implies that $M_\emptyset(\lambda)$ is irreducible if and only if $\lambda(c) \neq 0$ [10], [7].

Suppose that $\lambda(c) = 0$ and denote $\pi(\lambda) = \{\beta \in \pi \mid (\lambda, \beta) = 0\}$. Let $0 \neq v \in M_\emptyset(\lambda)$. Then $N = U(\mathfrak{G})(\mathfrak{G}_-(\pi(\lambda)) \oplus G_-)v$ is a proper submodule of $M_\emptyset(\lambda)$, and we consider the $\mathfrak{G}$-module $M'(\lambda) = M_\emptyset(\lambda)/N$. Clearly, $L_\emptyset(\lambda)$ is the unique irreducible quotient of $M'(\lambda)$. Moreover, we have the following result.

Corollary 3.3 (cf. [10], Proposition 6.2 and [7], Theorem 1, (ii))**.** *The module $M'(\lambda)$ is irreducible.*

Proof. Follows immediately from Proposition 3.2 with $n_0 = 0$, $\epsilon = \pi$ and $\mu = \lambda$. □

Assume that $\pi(\lambda) = \emptyset$. Then the maximal submodule of $M_\emptyset(\lambda)$ is generated by $M_1 = \sum_{n=1}^{\infty} M_\emptyset(\lambda)_{\lambda - n\delta}$ by Corollary 3.3. Moreover, any submodule $M \subset M_\emptyset(\lambda)$ is generated by $M \cap M_1$. A more general statement (Lemma 5.1) will be proved in section 5. Clearly, $M_\emptyset(\lambda)$ has a local composition series with all irreducible subquotients isomorphic to $M'(\lambda - n\delta)$, $n \in \mathbb{Z}_+$, and $[M_\emptyset(\lambda) : M'(\lambda - n\delta)]$ does not depend on the choice of a local composition series for any $n \in \mathbb{Z}_+$.

Proposition 3.4 ([7], Theorem 2)**.** *Let $\lambda \in \mathfrak{H}^*$, $\lambda(c) = 0$ and $\pi(\lambda) = \emptyset$. Then $\mathrm{Hom}_{\mathfrak{G}}(M_\emptyset(\mu), M_\emptyset(\lambda)) \neq 0$ if and only if $\mu = \lambda - n\delta$ for some $n \in \mathbb{Z}_+$, and moreover $\dim \mathrm{Hom}_{\mathfrak{G}}(M_\emptyset(\lambda - n\delta), M_\emptyset(\lambda)) = [M_\emptyset(\lambda) : M'(\lambda - n\delta)] = \dim M_\emptyset(\lambda)_{\lambda - n\delta}$.*

Remark 3.5. If $\lambda \in \mathfrak{H}^*$, $\lambda(c) = 0$ and $\pi(\lambda) = \pi$, then $L_\emptyset(\lambda)$ is the trivial one-dimensional module and the maximal submodule of $M_\emptyset(\lambda)$ has irreducible subquotients which are not of type $L_\emptyset(\mu)$ [7].

4. Verma type modules of zero level

From now on we assume that $X \neq I \cup \emptyset$. In this section we begin a study of Verma type modules $M_X(\lambda)$ with $\lambda(c) = 0$. Consider a subspace $\bar{M} = U(\bar{G}_-)\bar{G}_- v_\lambda$ of $M_X(\lambda)$.

Lemma 4.1. *$U(\mathfrak{G})\bar{M}$ is a proper submodule of $M(\lambda)$.*

Proof. If $U(\mathfrak{G})\bar{M} = M_X(\lambda)$ then, by the PBW Theorem, $v_\lambda = \sum_i u_i^- u_i^+ X_i v_\lambda$, where $u_i^- \in U(\mathfrak{G}_X^-)$, $u_i^+ \in U(\mathfrak{G}_X^+)\mathfrak{G}_X^+$, $X_i \in U(\bar{\mathfrak{G}}_-)\bar{G}_-$. But $[\mathfrak{G}_X^+, \bar{G}] \subset \mathfrak{G}_X^+ \oplus Z$. Thus $v_\lambda = \sum_i u_i^- X_i u_i^+ v_\lambda + \sum_i u_i^- [u_i^+, X_i] v_\lambda = 0$, and this contradiction completes the proof. □

Let $\nu : U(\mathfrak{G}) \to U(\mathfrak{G})/U(\mathfrak{G})\mathfrak{G}_X^+$ be the natural map. We will identify the elements of $U(\mathfrak{G}_X^-)$ with their images under ν. Also consider the natural map $\tilde{\nu} : U(\mathfrak{G}) \to U(\mathfrak{G})/U(\mathfrak{G})(\mathfrak{G}_X^+ \oplus \bar{G}_-)$ and the following decomposition of $U(\mathfrak{G}_X^-)$: $U(\mathfrak{G}_X^-) \simeq U(\mathfrak{G}'_-) \otimes_{\mathbb{C}} U(\mathbf{m}^-) \otimes_{\mathbb{C}} U(\tilde{\mathfrak{G}}_-)$. Then $\tilde{\nu}(U(\mathfrak{G}_X^-)) = U(\mathfrak{G}_X^-)/U(\mathfrak{G}_X^-)\bar{G}_- \simeq U(\mathfrak{G}'_-) \otimes_{\mathbb{C}} U(\mathfrak{G}_-^f) \otimes_{\mathbb{C}} U(\tilde{\mathfrak{G}}_-)$.

Denote $\tilde{M}(\lambda) = M_X(\lambda)/U(\mathfrak{G})\bar{M}$. Then $\tilde{M}(\lambda) \simeq U(\mathfrak{G}'_-) \otimes_{\mathbb{C}} U(\mathfrak{G}_-^f) \otimes_{\mathbb{C}} U(\tilde{\mathfrak{G}}_-)$ as a vector space. Let $\tilde{M}^f(\lambda) = \sum \tilde{M}(\lambda)_\mu, \dim \tilde{M}(\lambda)_\mu < \infty$. Clearly, $\tilde{M}(\lambda) \simeq M^f(\lambda)/U(\mathfrak{G}^f)\bar{M}$, and hence it is a Verma $\tilde{\mathfrak{G}}^f$–module with highest weight λ with respect to the triangular decomposition $\tilde{\mathfrak{G}}^f = \mathfrak{G}_-^f \oplus \mathfrak{H} \oplus \mathfrak{G}_+^f$ [13]. In particular, $\tilde{M}^f(\lambda) \simeq U(\mathfrak{G}_-^f)$ as a vector space. Consider the natural map $\tau : M_X(\lambda) \to \tilde{M}(\lambda)$ and let $\tilde{v}_\lambda = \tau(v_\lambda)$.

Lemma 4.2. *Let $0 \neq v_0 \in \tilde{M}^f(\lambda)_\mu, \mu \in H^*$. A $\tilde{\mathfrak{G}}$-module $V = U(\tilde{\mathfrak{G}})v_0$ is irreducible if and only if $(\mu, \beta) \neq 0$ for all $\beta \in \tilde{\pi}$.*

Proof. Since $\tilde{\mathfrak{G}}_+ v_0 = \bar{G}v_0 = 0$, then V is isomorphic to $U(\tilde{\mathfrak{G}}_-)$ as a vector space, and the lemma follows from Proposition 3.2 if we let $\epsilon = \tilde{\pi}$ and $n_0 = 0$. □

Lemma 4.3. *Let $0 \neq u \in U(\mathfrak{G}_X^-)$, u homogeneous, and $ht'(u) > 1$. Then there exist $\phi \in \Delta'_+$, $n \in \mathbb{Z}_+$ and $y \in \mathfrak{G}_{\phi - n\delta}$ such that $\tilde{\nu}(yu) \neq 0$ and $ht'(\tilde{\nu}(yu)) \neq 0$.*

Proof. Using the PBW Theorem we can write

$$u = \sum_k X^{\ell_{1k}}_{-\phi_{1k}+n_{1k}\delta} \dots X^{\ell_{s(k)k}}_{-\phi_{s(k)k}+n_{s(k)k}\delta} u_k,$$

where $ht'(\phi_{ik}) \neq 0$ for all i, k, $-\phi_{ik} + n_{ik}\delta \neq -\phi_{jk} + n_{jk}\delta$ if $i \neq j$ for all k; n_{ik}, $\ell_{ik} \in \mathbb{Z}$, $\ell_{ik} > 0$, and $u_k \in U(\mathfrak{G}_-^f \oplus \tilde{\mathfrak{G}}_-)$. By the assumption, $\sum_i ht'(\phi_{ik}) = ht'(u)$ for each k. Consider a subset $\Omega \subset \{\phi_{ik}\}$ consisting of all ψ such that $ht'(\psi) = \min_{i,k} ht'(\phi_{ik})$. We may assume that $\phi_{s(1)1} \in \Omega$, $ht'(\phi_{1k}) \geq \dots \geq ht'(\phi_{s(k)k})$ for all k, and that $-\phi_{s(1)1} + n_{s(1)1}\delta = -\phi_{ik} + n_{ik}\delta$ implies $i = s(k)$. If $\phi_{s(1)1} \in \Delta$ then by Lemma 3.1, (2) there exists $\phi \in \Delta'_+$ such that $ht'(\phi) = 1$ and $\phi - \phi_{s(1)1} \in \dot{\Delta}_-$. By Lemma 3.1, (4), we can choose sufficiently large $n \in \mathbb{Z}_+$ for which $\phi - \phi_{s(1)1} + (n_{s(1)1} - n)\delta \in -P(X)$, $n > |n_{ik}|$ for all i, k and $n > |u_k|_+$ for all k. For $0 \neq y \in \mathfrak{G}_{\phi - n\delta}$ it follows that $\tilde{\nu}(yX_{-\phi_{s(1)1}+n_{s(1)1}\delta}) \neq 0$, $\tilde{\nu}(yu) \neq 0$ and $ht'(\tilde{\nu}(yu)) \neq 0$. If $\frac{1}{2}\phi_{s(1)1} \in \Delta$ then $\tilde{\nu}(X_{\frac{1}{2}\phi_{s(1)1}}u) \neq 0$, which completes the proof. □

Lemma 4.4. *Let $\psi \in \Delta'_+$, $ht'(\psi) = 1$, $n \in \mathbb{Z}$ and $0 \neq X_{-\psi+n\delta} \in \mathfrak{G}_{-\psi+n\delta}$. Then there exists $y \in U(\mathfrak{G})$ such that $0 \neq \tilde{\nu}(yX_{-\psi+n\delta}) \in U(\mathfrak{G}_-^f)$.*

Proof. If $ht^f(\psi) = 0$ then the statement follows from Lemma 3.1, (3). Let $ht^f(\psi) \neq 0$. By Lemma 3.1, (1) and (4) there exist $\phi \in \Delta'_+$ for which $\phi - \psi \in \dot{\Delta}_-^f$, and $t \in \mathbb{Z}_+$ such that $\phi - \psi + (n - t)\delta \in -P(X)$ and $\phi - t\delta \in \Delta$. Then for $0 \neq y \in \mathfrak{G}_{\phi - m\delta}$ we have $0 \neq \tilde{\nu}(yX_{-\psi+n\delta}) \in U(\mathfrak{G}_-^f)$, and the lemma follows. □

Proposition 4.5. *Let $0 \neq u \in U(\mathfrak{G}_X^-)$, u homogeneous, $ht'(u) \neq 0$ and $\tilde{\nu}(u) \neq 0$. Then there exists $y \in U(\mathfrak{G})$ such that $0 \neq \tilde{\nu}(yu) \in U(\mathfrak{G}_-^f) \otimes_{\mathbb{C}} U(\tilde{\mathfrak{G}}_-)$.*

Proof. Let $u = \sum_k X^{\ell_{ik}}_{-\phi_{1k}+n_{1k}\delta} \dots X^{\ell_{s(k)k}}_{-\phi_{s(k)k}+n_{s(k)k}\delta} u_k + u_0$, where $ht'(\phi_{ik}) \neq 0$ for all i, k, ϕ_{ik} are ordered as in the proof of Lemma 4.3, ℓ_{ik}, $n_{ik} \in \mathbb{Z}$, $\ell_{ik} > 0$, $u_k \in U(\mathfrak{G}_-^f) \otimes_{\mathbb{C}} U(\tilde{\mathfrak{G}}_-)$ and $u_0 \in U(\mathfrak{G})\bar{G}_-$. Suppose that $ht'(u) = \sum_i ht'(\phi_{ik}) = r$. Using Lemmas 4.3 and 4.4 and induction on r, we can choose $\psi_1, \dots, \psi_r \in \Delta'_+$ and sufficiently large $t_1, \dots, t_r$ such that $0 \neq \tilde{\nu}(y_r \dots y_1 u) \in U(\mathfrak{G}_-^f) \otimes_{\mathbb{C}} U(\tilde{\mathfrak{G}}_-)$, where $0 \neq y_i \in \mathfrak{G}_{\psi_i - t_i\delta}$, $i = 1, \dots, r$. □

Remark 4.6. Proposition 4.5 is also valid in the case when the element u is not homogeneous, i.e., $u = \sum_i c_i u_i$ where $u_i \in U(\mathfrak{G}_X^-)_{\eta_i}$, $\eta_i \in Q$ and $\eta_i \neq \eta_j$ if $i \neq j$.

Proposition 4.7. *Let $0 \neq v \in \tilde{M}(\lambda)_\mu$, $\mu \in \mathfrak{H}^*$, and $(\lambda, \alpha) \neq 0$ for all $\alpha \in \tilde{\pi}$. Then there exists $y \in U(\mathfrak{G})$ such that $0 \neq yv \in \tilde{M}^f(\lambda)$.*

Proof. Let $v = u\tilde{v}_\lambda$ where $u \in U(\mathfrak{G}_X^-)_{\mu-\lambda}$. By Proposition 4.5 there exists $y' \in U(\mathfrak{G})$ such that $0 \neq \tilde{\nu}(y'u) \in U(\mathfrak{G}_-^f) \otimes U(\tilde{\mathfrak{G}}_-)$ and thus $y'v = y'u\tilde{v}_\lambda = \tilde{\nu}(y'u)\tilde{v}_\lambda \neq 0$.

Using the PBW Theorem we can write $\tilde{\nu}(y'u) = \sum_k u_k^{(1)} u_k^{(2)}$, where $u_k^{(1)} \in U(\mathfrak{G}_-^f)$ and $u_k^{(2)}$ are the monomials in $U(\tilde{\mathfrak{G}}_-)$. Let

$$u_k^{(2)} = X_{-\phi_{1k}+n_{1k}\delta}^{\ell_{1k}} \dots X_{-\phi_{S(k)k}+n_{s(k)k}\delta}^{\ell_{s(k)k}},$$

where $\phi_{ik} \in Q_{\tilde{\pi}}^+, n_{ij}, \ell_{ij} \in \mathbb{Z}, \ell_{ij} > 0$. Among all monomials $u_k^{(2)}$ consider those with smallest $|u_k^{(2)}|$ and denote them by $\tilde{u}_k^{(2)}$. Thus $\tilde{\nu}(y'u) = \sum_n u_n^{(1)} \tilde{u}_n^{(2)} + \sum_t u_t^{(1)} u_t^{(2)}$. Note that $\tilde{ht}(\tilde{u}_n^{(2)}) = \tilde{ht}(u_t^{(2)})$ for all n, t and $\tilde{u}_n^{(2)} \in U(\tilde{\mathfrak{G}}_-)_\eta$ for a single $\eta \in Q$ for all n. By Lemma 4.2 there exists $z \in U(\tilde{\mathfrak{G}})_{-\eta}$ such that $z(\sum_n \tilde{u}_n^{(2)})\tilde{v}_\lambda \in \mathbb{C}^* \tilde{v}_\lambda$ and hence $0 \neq z(\sum_n u_n^{(1)} \tilde{u}_n^{(2)}) v_\lambda \in \tilde{M}^f(\lambda)$. Since $z\tilde{v}_\lambda = 0$ and $\nu(z(\sum_t u_t^{(1)} u_t^{(2)})) \in U(\bar{G}_+)\bar{G}_+$, this implies that $0 \neq zy'v = zy'u\tilde{v}_\lambda = z\tilde{\nu}(y'u)\tilde{v}_\lambda = z(\sum_n u_n^{(1)} \tilde{u}_n^{(2)})\tilde{v}_\lambda \in \tilde{M}^f(\lambda)$. We complete the proof by setting $y = zy'$. □

Now we are in a position to prove the criterion of irreducibility for modules $\tilde{M}(\lambda)$.

Theorem 4.8. *Let $\lambda(c) = 0$. The module $\tilde{M}(\lambda)$ is irreducible if and only if the following two conditions hold:*

1. *$\tilde{M}^f(\lambda)$ is an irreducible $\mathfrak{G}^f$-module.*
2. *$(\lambda, \alpha) \neq 0$ for all $\alpha \in \tilde{\pi}$.*

Proof. Assume $\tilde{M}(\lambda)$ is an irreducible $\mathfrak{G}$-module. If N^f is a proper $\mathfrak{G}^f$-submodule of $\tilde{M}^f(\lambda)$ then $U(\mathfrak{G})N^f$ is a proper $\mathfrak{G}$-submodule of $\tilde{M}(\lambda)$. Thus $N^f = 0$ and $\tilde{M}^f(\lambda)$ is an irreducible $\mathfrak{G}^f$-module. Suppose that $(\lambda, \alpha) = 0$ for some $\alpha \in \tilde{\pi}$. Since $[\mathfrak{G}_{\alpha+n\delta}, \mathfrak{G}_{-\alpha}] \subset \bar{G}$ for any $n \in \mathbb{Z} \setminus \{0\}$ and $[\mathfrak{G}_\alpha, \mathfrak{G}_{-\alpha}]\tilde{v}_\lambda = 0$, we have that $U(\mathfrak{G})\mathfrak{G}_{-\alpha}\tilde{v}_\lambda$ is a proper $\mathfrak{G}$-submodule of $\tilde{M}(\lambda)$, which again contradicts the irreducibility of $\tilde{M}(\lambda)$.

Conversely, suppose that conditions (1) and (2) of the theorem are satisfied. Let N be a non-zero $\mathfrak{G}$-submodule of $\tilde{M}(\lambda)$ and $0 \neq v \in N$. Then by Proposition 4.7 there exists $y \in U(\mathfrak{G})$ such that $0 \neq yv \in \tilde{M}^f(\lambda)$. Since $\tilde{M}^f(\lambda)$ is an irreducible $\mathfrak{G}^f$-module, yv generates $\tilde{M}(\lambda)$. Thus, $N = \tilde{M}(\lambda)$ and $\tilde{M}(\lambda)$ is irreducible. □

5. Structure of modules $M_X(\lambda)$ when $\tilde{M}(\lambda)$ is irreducible

Let $\lambda \in \mathfrak{H}^*$, $\lambda(c) = 0$. In this section we will assume that $\tilde{M}(\lambda)$ is an irreducible $\mathfrak{G}$-module, i.e. it satifies the conditions (1) and (2) of Theorem 4.8. If N is a submodule of $M_X(\lambda)$ we will denote $[N] = N \cap \bar{M}$.

Lemma 5.1. *Suppose that $\tilde{M}(\lambda)$ is an irreducible $\mathfrak{G}$-module. Then for every non-zero submodule N of $M_X(\lambda)$, $[N] \neq 0$ and N is generated by $[N]$.*

Proof. Let N be a non-zero proper submodule of $M_X(\lambda)$ and $0 \neq v \in N$. Then $v = uv_\lambda$ for some $u \in U(\mathfrak{G}_X^-)$. Using the PBW Theorem we can write u as the following linear combination:

$$u = \sum_{k \in K_0} a_k u_k^{(1)} u_k^{(2)} u_k^{(3)} u_k^{(4)}, \tag{5.1}$$

where $a_k \in \mathbb{C}^*$ and $u_k^{(i)}$ are monomials such that $u_k^{(1)} \in U(\mathfrak{G}'_-)$, $u_k^{(2)} \in U(\mathfrak{G}_-^f)$, $u_k^{(3)} \in U(\tilde{\mathfrak{G}}_-)$, $u_k^{(4)} \in U(\bar{G}_-)$. We will also assume that u is homogeneous. If

$u_k^{(4)} = 0$ for at least one $k \in K_0$, then $\tilde{\nu}(u) \neq 0$ and by Theorem 4.8 $\tilde{M}(\lambda) \simeq \tau(N)$ where $\tau : M_X(\lambda) \to \tilde{M}(\lambda)$. Thus $v_\lambda \in N$ and $N = M$, which contradicts our assumption. Therefore we can assume that $u_k^{(4)} \neq 0$ for all $k \in K_0$. Denote by $\ell(u)$ the number of different $u_k^{(4)}$, $k \in K_0$, in (5.1). Suppose that $\ell(u) = 1$. Then

$$u = \left(\sum_{k \in K_0} a_k u_k^{(1)} u_k^{(2)} u_k^{(3)} \right) u^{(4)}.$$

Consider an element $u' = \sum_{k \in K_0} a_k u_k^{(1)} u_k^{(2)} u_k^{(3)} \in U(\mathfrak{G}'_- \oplus \mathfrak{G}^f_- \oplus \tilde{\mathfrak{G}}_-)$. By Theorem 4.8 there exists $y \in U(\mathfrak{G}^f_X)$ such that $yu'v_\lambda = \nu(yu')v_\lambda \in \mathbb{C}^* v_\lambda$, and hence $\nu(yu') \in S(\mathfrak{H})$. Then we have $yuv_\lambda = yu'u^{(4)}v_\lambda = u^{(4)}\nu(yu')v_\lambda \in \mathbb{C}^* u^{(4)} v_\lambda$, which implies $u^{(4)}v_\lambda \in [N]$ and $v \in U(\mathfrak{G})[N]$. We conclude that $N = U(\mathfrak{G})[N]$.

Suppose now that $\ell(u) > 1$. Applying the same procedure as in the proof of Proposition 4.5 (see also Remark 4.6) we find an element $y_1 \in u(\mathfrak{G})$ such that $y_1 u v_\lambda = (\sum_m \tilde{a}_m \tilde{u}_m^{(2)} \tilde{u}_m^{(3)} \tilde{u}_m^{(4)}) v_\lambda \neq 0$, where $\tilde{u}_m^{(2)} \in U(\mathfrak{G}^f_-)$, $\tilde{u}_m^{(3)} \in U(\tilde{\mathfrak{G}}_-)$, $\tilde{u}_m^{(4)} \in \mathfrak{U}(\bar{G}_-)$ and for each m there exists $k \in K_0$ such that $\tilde{u}_m^{(4)} = u_k^{(4)}$. Note that $(\sum_m \tilde{a}_m \tilde{u}_m^{(2)} \tilde{u}_m^{(3)}) v_\lambda \in \tilde{M}^f(\lambda)$.

Let $u_m^{2,3} = \tilde{u}_m^{(2)} \tilde{u}_m^{(3)}$ for all m and $d = \max_m ||u_m^{2,3}||$. Consider the element

$$w = \sum_m \tilde{a}_m \tilde{u}_m^{(2)} \tilde{u}_m^{(3)} = \sum_{m : ||u_m^{2,3}|| = d} \tilde{a}_m u_m^{2,3} + \sum_{m : ||u_m^{2,3}|| \neq d} \tilde{a}_m u_m^{2,3}.$$

Since $M_X^f(\lambda)$ is irreducible by Theorem 4.8, there exists an element $y_2 \in U(\mathfrak{G}^f_+)$ such that

$$y_2 (\sum_{m : ||u_m^{2,3}|| = d} \tilde{a}_m u_m^{2,3}) v_\lambda \in \mathbb{C}^* v_\lambda.$$

Also, note that $y_2(\sum_{m:||u_m^{2,3}||\neq d} \tilde{a}_m u_m^{2,3}) v_\lambda = 0$. Thus $y_2 w \in \mathbb{C}^* v_\lambda$ and

$$y_2 y_1 v = y_2 y_1 u v_\lambda = (\sum_{\ell \in \mathfrak{L}} b_\ell \tilde{\tilde{u}}_\ell) v_\lambda \neq 0,$$

where for each $\ell \in \mathfrak{L}$ there exists $k \in K_0$ such that $\tilde{\tilde{u}}_\ell = u_k^{(4)}$ and $\tilde{\tilde{u}}_m \neq \tilde{\tilde{u}}_n$ if $m \neq n$. We conclude that $0 \neq y_2 y_1 v \in N \cap \bar{M}$, and hence $[N] \neq 0$.

Suppose that $b_t \neq 0$. Then we can write u in the form

$$u = \hat{u}_t \tilde{\tilde{u}}_t + \sum_{\ell \in \mathfrak{L} \setminus \{t\}} \hat{u}_\ell \tilde{\tilde{u}}_\ell + \sum_{k \in K} \hat{u}_k u_k^{(4)},$$

where $\hat{u}_m \in U(\mathfrak{G}^f_- \oplus \tilde{\mathfrak{G}}_- \oplus \mathfrak{G}'_-)$, $\tilde{\tilde{u}}_\ell \neq u_k^{(4)}$ for all ℓ, k and $u_m^{(4)} \neq u_n^{(4)}$ if $m \neq n$. Then $\ell(u) = |\mathfrak{L}| + |K|$. Consider the element

$$\zeta = u - \frac{1}{b_t} \hat{u}_t (\sum_\ell b_\ell \tilde{\tilde{u}}_\ell) = \sum_{\ell \in \mathfrak{L} \setminus \{t\}} (\hat{u}_\ell - \frac{b_\ell}{b_t} \hat{u}_t) \tilde{\tilde{u}}_\ell + \sum_{k \in K} \hat{u}_k u_k^{(4)}.$$

Then $\zeta v_\lambda = v - \frac{1}{b_t} \hat{u}_t y_2 y_1 v \in N$. If $\zeta v_\lambda = 0$ we obtain that $K = \emptyset$, $\hat{u}_\ell = \frac{b_\ell}{b_t} \hat{u}_t$ for all $\ell \in \mathfrak{L}$, $u = \hat{u}_t \tilde{\tilde{u}}_t + \sum_{\ell \in \mathfrak{L} \setminus \{t\}} \frac{b_\ell}{b_t} \hat{u}_t \tilde{\tilde{u}}_\ell = \frac{1}{b_t} \hat{u}_t (\sum_{\ell \in \mathfrak{L}} b_\ell \tilde{\tilde{u}}_\ell)$, and $v = u v_\lambda \in U(\mathfrak{G})[N]$. Lemma 5.1 is proved.

Suppose now that $\zeta v_\lambda \neq 0$. Since $\ell(\zeta) = |K| + |\mathfrak{L} \setminus \{t\}| = \ell(u) - 1$, we can apply induction on $\ell(u)$ and conclude that $\tilde{\tilde{u}}_\ell v_\lambda \in [N]$, $\ell \in \mathfrak{L} \setminus \{t\}$ and $u_k^{(4)} v_\lambda \in [N]$,

$k \in K$. Hence $u_k^{(4)} v_\lambda \in [N]$ for all $k \in K_0$. It implies that $uv_\lambda \in U(\mathfrak{G})[N]$ and $N = U(\mathfrak{G})[N]$. This completes the proof of the lemma. □

Let $N \subset M_X(\lambda)$. It follows from Lemma 5.1 that N has a local composition series with all irreducible quotients isomorphic to $\tilde{M}(\lambda - m\delta)$, $m \in \mathbb{Z}_+$. Moreover the number $[N : \tilde{M}(\lambda - m\delta)]$ does not depend on the choice of a local composition series for any $m \in \mathbb{Z}_+$. The following statement is a generalization of Proposition 3.4.

Proposition 5.2. *Suppose $\lambda \in \mathfrak{H}^*$, $\lambda(c) = 0$, $m \in \mathbb{Z}_+$ and $\tilde{M}(\lambda)$ is irreducible. If $N \subset M_X(\lambda)$ and $\mu \in \mathfrak{H}^*$, then*

1. *$\tilde{M}(\lambda - m\delta)$ is an irreducible $\mathfrak{G}$-module.*
2. *$[N : \tilde{M}(\lambda - m\delta)] = \dim(\bar{M} \cap N_{\lambda - m\delta})$.*
3. *$\mathrm{Hom}_{\mathfrak{G}}(M_X(\mu), M_X(\lambda)) \neq 0$ if and only if $\mu = \lambda - n\delta$ for some $n \geq 0$ and $\dim \mathrm{Hom}_{\mathfrak{G}}(M_X(\lambda - n\delta), M_X(\lambda)) = \dim(\bar{M} \cap M_X(\lambda)_{\lambda - n\delta})$.*

Proof. Statement 1 follows from Theorem 4.8, while 2 and 3 follow from Proposition 5.2. □

6. Irreducible quotients of $M_X(\lambda)$ (case of irreducible $\tilde{M}^f(\lambda)$)

Let $\lambda \in \mathfrak{H}^*$, $\lambda(c) = 0$. By Theorem 4.8, if $\tilde{M}^f(\lambda)$ is an irreducible $\mathfrak{G}^f$-module and $(\lambda, \alpha) \neq 0$ for all $\alpha \in \tilde{\pi}$, then $L_X(\lambda) \simeq \tilde{M}(\lambda) \simeq U(\mathfrak{G}'_-) \otimes_{\mathbb{C}} U(\tilde{\mathfrak{G}}_-) \otimes_{\mathbb{C}} \tilde{M}^f(\lambda)$ as vector spaces. In this section we consider the case when $\tilde{M}^f(\lambda)$ is an irreducible $\mathfrak{G}^f$-module and there exists at least one $\alpha \in \tilde{\pi}$ such that $(\lambda, \alpha) = 0$, which implies that $\tilde{M}(\lambda)$ is no longer irreducible.

Set $\mathfrak{A} = \mathfrak{G}_-(\tilde{\pi}(\lambda))$, where

$$\tilde{\pi}(\lambda) = \{\alpha \in \tilde{\pi} | (\lambda, \alpha) = 0\}, \hat{U} = U(\mathfrak{G}'_-) \otimes (U(\tilde{\mathfrak{G}}_-)/U(\tilde{\mathfrak{G}}_-)\mathfrak{A}), \mathfrak{B} = U(\mathfrak{G})\mathfrak{A}\tilde{v}_\lambda$$

and $\hat{M}(\lambda) = \tilde{M}(\lambda)/\mathfrak{B}$. Let $\hat{\nu} : U(\mathfrak{G}) \to \hat{U} \otimes_{\mathbb{C}} U(\mathfrak{G}^f_-)$ and $\hat{\tau} : \tilde{M}(\lambda) \to \hat{M}(\lambda)$ be natural maps and $\hat{v}_\lambda = \hat{\tau}(\tilde{v}_\lambda)$. We will identify the elements of $\hat{U}$ with their representatives in $U(\mathfrak{G}'_-) \otimes U(\tilde{\mathfrak{G}}_-)$.

Theorem 6.1. *$L_X(\lambda) \simeq \hat{U} \otimes_{\mathbb{C}} \tilde{M}^f(\lambda)$ as vector spaces.*

Proof. One can easily see that $\mathfrak{B}$ is a proper submodule of $\tilde{M}(\lambda)$. But $\mathfrak{B} \simeq \hat{U} \otimes_{\mathbb{C}} U(\mathfrak{A})\mathfrak{A} \otimes_{\mathbb{C}} \tilde{M}^f(\lambda)$, and thus $\hat{M}(\lambda) \simeq \hat{U} \otimes_{\mathbb{C}} \tilde{M}^f(\lambda)$ as vector spaces. We will show that the $\mathfrak{G}$-module $\hat{M}(\lambda)$ is irreducible. Let $N \subset \hat{M}(\lambda)$ and $0 \neq v \in N$, $v = u\hat{v}_\lambda$, where $u \in U(\mathfrak{G}'_-) \otimes U(\mathfrak{G}^f_-) \otimes (U(\tilde{\mathfrak{G}}_-)/U(\tilde{\mathfrak{G}}_-)\mathfrak{A})$. If $ht'(u) \neq 0$, it follows from the proof of Lemma 4.4 that there exists $y \in U(\mathfrak{G})$ such that $0 \neq \hat{\nu}(yu) \in U(\mathfrak{G}^f_-) \otimes_{\mathbb{C}} (U(\tilde{\mathfrak{G}}_-)/U(\tilde{\mathfrak{G}}_-)\mathfrak{A})$.

Thus we can assume that $ht'(u) = 0$. If $\tilde{ht}(u) \neq 0$, then following the proof of Proposition 4.7 we find an element $y_1 \in U(\mathfrak{G})$ such that $0 \neq \hat{\nu}(y_1 u) \in U(\mathfrak{G}^f_-)$. Hence we may assume that $u \in U(\mathfrak{G}^f_-)$ and $v = u\hat{v}_\lambda \in \tilde{M}^f(\lambda)$. But $\tilde{M}^f(\lambda)$ is an irreducible $\mathfrak{G}^f$–module, which implies that $N = \hat{M}(\lambda)$. The theorem is proved. □

Remark 6.2. Let $\alpha \in \tilde{\pi}$ and $(\lambda, \alpha) = 0$. The structure of the module $M_X(\lambda)$ in this case is quite mysterious. For example, $M_X(\lambda)$ has irreducible subquotients which are not of type $L_X(\mu)$ [7].

7. Irreducible quotients of $M_X(\lambda)$ (case of reducible $\tilde{M}^f(\lambda)$)

In this section we will assume that the $\mathfrak{G}^f$–module $\tilde{M}^f(\lambda)$ is reducible. Thus $L_X(\lambda)$ is the unique irreducible quotient of $\hat{M}(\lambda)$ and $L^f(\lambda)$ is the unique irreducible quotient of $\tilde{M}^f(\lambda)$.

Remark 7.1. One would expect that $L_X(\lambda) \simeq \hat{U} \otimes L_X^f(\lambda)$, but this is not always the case. For example, if $L_X^f(\lambda) = \mathbb{C}v$ and $(\lambda, \beta) = 0$ for some $\beta \in \pi'$, then $U(\mathfrak{G})(\mathfrak{G}_{-\beta} \otimes v)$ is a proper submodule of $\hat{U} \otimes \mathbb{C}v$ and thus $L_X(\lambda) \not\simeq \hat{U} \otimes \mathbb{C}v$.

Conjecture 7.2. *If $\lambda \in \mathfrak{H}^*$, $\lambda(c) = 0$ and $\dim L_X^f(\lambda) > 1$, then $L_X(\lambda) \simeq \hat{U} \otimes L_X^f(\lambda)$.*

We will prove this conjecture for λ in "general position".

Definition 7.3. Let $\lambda \in \mathfrak{H}^*$, $\lambda(c) = 0$. We will say that λ is in *general position* if $(\mu, \beta) \neq 0$ for all $\mu \in \lambda + Q_{\pi^f}$ and any $\beta \in \pi' \setminus \tilde{\pi}$.

Example 7.4. Any $\lambda \in \mathfrak{H}^*$ such that $\lambda(c) = 0$ and $\frac{2(\lambda,\beta)}{(\beta,\beta)} \notin \mathbb{Z}$ for all $\beta \in \pi' \setminus \tilde{\pi}$ is in general position.

Let $u \in \hat{U} \otimes U(\mathfrak{G}_-^f)$ be a homogeneous element. We will say that u is represented in a normal form if $u = \sum_{k \in K} u_k u_k^f$, where $u_k \in \hat{U}$, $u_k^f \in U(\mathfrak{G}_-^f)$, u_k are linearly independent and $u_i^f \notin \mathbb{C}^* u_j^f$ if $i \neq j$. Obviously, if $|K| > 1$ then u has many different normal forms.

Using the PBW Theorem we can choose a basis in $U(\mathfrak{G}_-^f)$ consisting of the monomials $u^{(\delta)} u^{(f)}$ where $u^{(\delta)} \in U(G_-) \cap U(\mathfrak{G}_-^f)$ and $u^{(f)}$ contains no elements of $U(G_-)$. Thus any element $u \in U(\mathfrak{G}_-^f)$ can be written uniquely as a linear combination of certain monomials $u_i^{(\delta)} u_i^{(f)}$. Suppose that $u \in U(\mathfrak{G}_-^f)_{\eta + m\delta}$, $u_i^{(f)} \in U(\mathfrak{G}_-^f)_{\eta_i + m_i\delta}$ and $\eta, \eta_i \in Q, m, m_i \in \mathbb{Z}$ for all i. Then we set $d(u) = \max_i |m_i|$ and $\bar{d}(u) = |m| - d(u)$. With each homogeneous element $u \in \hat{U} \otimes U(\mathfrak{G}_-^f)$ represented in a normal form $u = \sum_{k \in K} u_k u_k^f$ we will associate $D(u) = \max_{k \in K} d(u_k^f)$ and the set $S(u) = \{k \in K \mid d(u_k^f) = D(u)\}$. Note that $D(u)$ is independent of the choice of a normal form and depends only of u. On the other hand the set $S(u)$ is determined by a given normal form.

Let $\mathfrak{H}' = \mathfrak{H} \cap \mathfrak{G}(\pi')$, $G' = G \cap \mathfrak{G}(\pi')$. Consider the following decomposition of $U(\mathfrak{G}(\pi')) : U(\mathfrak{G}(\pi')) = U(\mathfrak{H}' \oplus G') \oplus (U(\mathfrak{G}(\pi'))\mathfrak{G}_+(\pi') + \mathfrak{G}_-(\pi')U(\mathfrak{G}(\pi')))$, and the corresponding projection $\nu_0 : U(\mathfrak{G}(\pi')) \to U(\mathfrak{H}' \oplus G')$.

Proposition 7.5. *Let λ, $\mu \in \mathfrak{H}^*$, $\lambda(c) = 0$ and λ is in general position, $N \subset \hat{M}(\lambda)$, $0 \neq u\hat{v}_\lambda \in N_\mu$ and $u = \sum_{k \in K} u_k u_k^f$ is a normal form of u such that $ht^f(u_k) = 0$ for all $k \in K$. Then $u_k^f \hat{v}_\lambda \in N$ for all $k \in K$.*

Proof. Denote $v = u\hat{v}_\lambda$. Assume that $u_k \in \hat{U}_{\eta_k}$ and set $\mu_k = \mu - \eta_k$, $\eta_k \in Q$, $k \in K$. Then $(\mu_k, \beta) \neq 0$ for any $\beta \in \pi'$ and $k \in K$, since λ is in "general position". We will prove the statement by induction on $|K|$, $D(u)$ and $|S(u)|$ simultaneously.

Step 1. Let $|K| = 1$. Then $u = u' u^f$, where u' is a homogeneous element of $\hat{U}$ and $ht^f(u') = 0$. Consider a $U(\mathfrak{G}_-(\pi'))$-module $V = U(\mathfrak{G}_-(\pi'))u^f \hat{v}_\lambda$ with $\mathfrak{G}_+(\pi')u^f \hat{v}_\lambda = 0$ and $\mathfrak{G}_{-\beta}(u^f \hat{v}_\lambda) = u^f \mathfrak{G}_{-\beta} \hat{v}_\lambda = 0$ for all $\beta \in \tilde{\pi}(\mu)$. Obviously, $v \in N \cap V$, and it follows from Proposition 3.2 that $u^f \hat{v}_\lambda \in U(\mathfrak{G}(\pi'))v \subset N$.

Step 2. Suppose now that $|K| > 1$ and $D(u) = 0$. Consider $\tilde{K} = \{k \in K \mid \bar{d}(u_k^f) \leq \bar{d}(u_j^k)$ for all $j \in K\}$. Let $k_0 \in \tilde{K}$. Then by Step 1 there exists $y \in U(\mathfrak{G}(\pi'))$ for which $yu_{k_0}u_{k_0}^f\hat{v}_\lambda = a_{k_0}u_{k_0}^f\hat{v}_\lambda$, $a_{k_0} \in \mathbb{C}^*$. Since u is homogeneous and $ht^f(u_k) = 0$ for all k, we have $\nu_0(yu_k) \in U(\mathfrak{H}) \otimes U(G)_0$, $k \in \tilde{K} \setminus \{k_0\}$ and $\nu_0(yu_k) \in U(\mathfrak{H}) \otimes_{\mathbb{C}} U(G)_0G_+$, $k \in K \setminus \tilde{K}$. As $d(u_k^f) = 0$ for all $k \in K$, we conclude that $yu_ku_k^f\hat{v}_\lambda = a_ku_k^f\hat{v}_\lambda$, $a_k \in \mathbb{C}$, $k \in \tilde{K} \setminus \{k_0\}$, and $yu_ku_k^f\hat{v}_\lambda = 0$, $k \in K \setminus \tilde{K}$. Hence $yv = \sum_{k\in K} yu_ku_k^f\hat{v}_\lambda = \sum_{k\in K}\nu_0(yu_k)u_k^f\hat{v}_\lambda = \sum_{k\in\tilde{K}} a_ku_k^f\hat{v}_\lambda$. Suppose that $\sum_{k\in\tilde{K}} a_ku_k^f = 0$. Then $u_{k_0}^f = \sum_{k\in K\setminus\{k_0\}} \frac{a_k}{a_{k_0}}u_k^f$ and $u = \sum_{k\in K\setminus\{k_0\}} u_ku_k^f + u_{k_0}u_{k_0}^f = \sum_{k\in\tilde{K}\setminus\{k_0\}}(u_k + \frac{a_k}{a_{k_0}}u_{k_0})u_k^f + \sum_{k\in K\setminus\tilde{K}} u_ku_k^f$. Applying induction on $|K|$, we conclude that $u_k^f\hat{v}_\lambda \in N$ for all $k \in K \setminus \{k_0\}$, and thus $u_k^f\hat{v}_\lambda \in N$ for all $k \in K$. Suppose now that $\sum_{k\in\tilde{K}} a_ku_k^f \neq 0$. Then $yv \neq 0$ and $0 \neq a_{k_0}v - u_{k_0}yv = (a_{k_0}u - u_{k_0}yu)\hat{v}_\lambda = u'\hat{v}_\lambda \in N$, where $u' = \sum_{k\in\tilde{K}\setminus\{k_0\}}(a_{k_0}u_k - a_ku_{k_0})u_k^f + \sum_{k\in K\setminus\tilde{K}} a_{k_0}u_ku_k^f$ is represented in a normal form. The induction on $|K|$ implies that $u_k^f\hat{v}_\lambda \in N$ for all $k \in K \setminus \{k_0\}$, and therefore $u_k^f\hat{v}_\lambda \in N$ for all $k \in K$.

Step 3. Now let $|K| > 1$, $D(u) > 0$ and $|S(u)| = 1$. Suppose that $S(u) = \{k_0\}$ and consider an element $y \in U(\mathfrak{G}(\pi'))$ as in Step 2. Let $yu_{k_0}u_{k_0}^f\hat{v}_\lambda = a_{k_0}u_{k_0}^f\hat{v}_\lambda$, $a_{k_0} \in \mathbb{C}^*$. Since $|S(u)| = 1$, we obtain $yv = yu\hat{v}_\lambda = a_{k_0}u_{k_0}^f\hat{v}_\lambda + \sum_{k\in K\setminus\{k_0\}}\nu_0(yu_k)u_k^f\hat{v}_\lambda \neq 0$ and $N \ni a_{k_0}v - u_{k_0}yv = u'\hat{v}_\lambda$, where

$$u' = \sum_{k\in K\setminus\{k_0\}} a_{k_0}u_ku_k^f - \sum_{k\in K\setminus\{k_0\}} u_{k_0}\nu_0(yu_k)u_k^f.$$

If $\eta_k = \eta_{k_0}$ then $\nu_0(yu_k) \in S(\mathfrak{H}) \otimes U(G)_0$; otherwise

$$\nu(yu_k) \in S(\mathfrak{H}) \otimes (U(G)(G_- \oplus G_+)).$$

Since the u_k are linearly independent, this implies that $0 \neq u'\hat{v}_\lambda \in N$ and $D(u') < D(u)$. Rewriting u' in a normal form if necessary, we can apply induction on $D(u)$. Hence $u_k^f\hat{v}_\lambda \in N$ for all $k \in K\setminus\{k_0\}$, implying $u_k^f\hat{v}_\lambda \in N$ for all $k \in K$ by induction on $|K|$.

Step 4. Suppose now that $|K| > 1$, $D(u) > 0$ and $|S(u)| > 1$. Denote $\tilde{K} = \{k \in S(u) | \bar{d}(u_k^f) \leq \bar{d}(u_j^f)$ for all $j \in S(u)\}$. Let $k_0 \in \tilde{K}$. Note that $\mu_k = \mu_{k_0}$ for all $k \in \tilde{K}$. As in Step 2, consider an element $y \in U(\mathfrak{G}(\pi'))$ for which $\nu_0(yu_{k_0}) \in S(\mathfrak{H}) \otimes U(G)_0$ and $yu_{k_0}u_{k_0}^f\hat{v}_\lambda = a_{k_0}u_{k_0}^f\hat{v}_\lambda$, $a_{k_0} \in \mathbb{C}^*$. Then $N \ni a_{k_0}v - u_{k_0}yv = (\sum_{k\in K\setminus\{k_0\}} a_{k_0}u_ku_k^f - \sum_{k\in K\setminus\{k_0\}} u_{k_0}\nu_0(yu_k)u_k^f)\hat{v}_\lambda$, $\nu_0(yu_k) \in S(\mathfrak{H}) \otimes U(G)_0$ if $\mu_k = \mu_{k_0}$, and $\nu_0(yu_k) \in S(\mathfrak{H}) \otimes (U(G)(G_+ \oplus G_-))$ otherwise. Clearly, $\hat{\nu}(yu_ku_k^f) = a_ku_k^f + u_k'$, $a_k \in \mathbb{C}$, $u_k' \in U(\mathfrak{G}_-^f)$ for $k \in \tilde{K} \setminus \{k_0\}$, and $\hat{\nu}(yu_ku_k^f) = u_k' \in U(\mathfrak{G}_-^f)$ for $k \in K \setminus \tilde{K}$, where $d(u_k') < D(u)$ for all $k \in K \setminus \{k_0\}$. Set $u'' = \sum_{k\in K\setminus\{k_0\}} u_k'$, $d(u'') < D(u)$. Thus $a_{k_0}v - u_{k_0}yv = u'\hat{v}_\lambda \neq 0$, where

$$u' = \sum_{k\in\tilde{K}\setminus\{k_0\}}(a_{k_0}u_k - a_ku_{k_0})u_k^f + \sum_{k\in K\setminus\tilde{K}} a_{k_0}u_ku_k^f - u_{k_0}u''.$$

If $u'' \notin \mathbb{C}^*u_k^f$ for all $k \in K \setminus \tilde{K}$, then u' is represented in a normal form, $|S(u')| < |S(u)|$ and induction on $|S(u)|$ implies $u_k^f\hat{v}_\lambda \in N$ for all $k \in K \setminus \{k_0\}$. By induction on $|K|$ we conclude that $u_k^f\hat{v}_\lambda \in N$ for all $k \in K$. Suppose now that $u'' = au_j^f$

for some $j \in K \setminus \tilde{K}$ and $a \in \mathbb{C}^*$. Then $u' = \sum_{k\in\tilde{K}\setminus\{k_0\}}(a_{k_0}u_k - a_k u_{k_0})u_k^f + \sum_{\substack{k\in K\setminus\tilde{K}\\ k\neq j}} a_{k_0}u_k u_k^f + (a_{k_0}u_j - au_{k_0})u_j^f$ is the normal form of u' and $|S(u')| < |S(u)|$. By induction on $|S(u)|$ and $|K|$ we conclude that $u_k^f \hat{v}_\lambda \in N$ for all $k \in K$, which completes the proof. □

Let $u \in \hat{U} \otimes U(\mathfrak{G}_-^f)$ be a homogeneous element. We will say that u is represented in the reduced normal form $u = \sum_i u_i u_i^f$ if $u_i = \sum_j a_{ij}\bar{u}_{ij}$ for each i, where $a_{ij} \in \mathbb{C}^*$, $\bar{u}_{ij}$ are monomials in $\hat{u}$ and $\bar{u}_{ij} = \bar{u}_{k\ell}$ only if $i = k$, $j = \ell$. Obviously, any non-zero homogeneous element of $\hat{U} \otimes U(\mathfrak{G}_-^f)$ has a reduced normal form.

Lemma 7.6. *Let $u = \sum_i u_i u_i^f$ be the reduced normal form of $u \in \hat{U} \otimes U(\mathfrak{G}_-^f)$, where $ht^f(u_i) \leq d$ for all i. Then there exists $y \in \mathfrak{G}'_+$ such that $\hat{u} = \hat{\nu}(yu) \neq 0$ and $\hat{u}$ has the reduced normal form $\hat{u} = \sum_j \hat{u}_j \hat{u}_j^f$, where $\hat{u}_j^f = u_i^f$ for at least one pair of indexes (i,j) and $ht^f(\hat{u}_j) \leq d-1$ for all j.*

Proof. Let $J = \{i \mid ht^f(u_i) = d\}$. For simplicity we may assume that all u_i are monomials in $\hat{U}$ and $u_i = X_{-\phi_{1i}+n_{1i}\delta} \dots X_{-\phi_{s(i)i}+n_{s(i)i}\delta} u_i'$, where $ht'(\phi_{ji}) \neq 0$ for all i,j, $\sum_j ht^f(\phi_{ji}) \leq d$, $0 < ht^f(\phi_{s(i)i} \leq \dots \leq ht^f(\phi_{2i}) \leq ht^f(\phi_{1i})$ and $ht^f(u_i') = 0$ for all i. We can also assume that u_i are numerated in such a way that $1 \in J$, $ht^f(\phi_{s(1)1}) \leq ht^f(\phi_{s(i)i})$ for any $i \in J$, if $ht^f(\phi_{s(1)1}) = ht^f(\phi_{s(i)i})$ then $ht'(\phi_{s(1)1}) \leq ht'(\phi_{s(i)i})$, and $-\phi_{s(1)1} + n_{s(1)1}\delta = -\phi_{ki} + n_{ki}\delta$ implies $k = s(i)$. Suppose that $\phi_{s(1)1} \in \Delta$. By Lemma 3.1, (5), there exists $\psi \in \Delta'_+$ such that $ht^f(\psi) = 1$ and $\psi - \phi_{s(1)1} \in \Delta'_-$. Then for large enough $m \in \mathbb{Z}_+$ and for $0 \neq y \in \mathfrak{G}_{\psi+m\delta}$ we have that $[y, u_i'] = 0$, $\hat{\nu}(yu_i^f) = 0$, $[y, u_i] \in \hat{U}$, $ht^f([y,u_i]) \leq d-1$ for $i \in J$ and $ht^f([y,u_j]) \leq d-2$ for $j \notin J$. Also note that a monomial $x = X_{-\phi_{11}+n_{11}\delta} \dots X_{-\phi_{s(1)-11}+n_{s(1)-11}\delta}[X_{\psi+m\delta}, X_{-\phi_{s(1)1}+n_{s(1)1}\delta}]u_1' \neq 0$, and it will appear in $[y, u_i]$ only for $i = 1$. Hence, $\hat{u} = \hat{\nu}(yu) = \hat{\nu}([y,u]) = \sum_i [y,u_i]u_i^f \neq 0$, and if $\hat{u} = \sum_j \hat{u}_j \hat{u}_j^f$ is the reduced normal form then $\hat{u}_k$ contains x for some k and $\hat{u}_k^f = u_1^f$. If $\phi_{s(1)1} \notin \Delta$ then $\frac{1}{2}\phi_{s(1)1} \in \Delta$, $\hat{\nu}(X_{\frac{1}{2}\phi_{s(1)1}}u) \neq 0$, and the same arguments as above complete the proof. □

If N is a $\mathfrak{G}$-submodule of $\hat{M}(\lambda)$, we set $N^f = \sum_{\mu\in\mathfrak{H}^*} N_\mu$, $0 < \dim N_\mu < \infty$. Clearly, N^f is isomorphic to a $\mathfrak{G}^f$-submodule of $\tilde{M}^f(\lambda)$.

Theorem 7.7. *Let $\lambda \in \mathfrak{H}^*$, $\lambda(c) = 0$ and λ is in general position.*

1. *If $N \subset \hat{M}(\lambda)$ then $N \simeq \hat{U} \otimes_{\mathbb{C}} N^f$ as vector spaces.*
2. *$L_X(\lambda) \simeq \hat{U} \otimes_{\mathbb{C}} L^f(\lambda)$ as vector spaces.*

Proof. Let $0 \neq v \in N$. Without loss of generality we can assume that v is a weight vector. Then $v = u\hat{v}_\lambda$ for some homogeneous element $u \in \hat{U} \otimes U(\mathfrak{G}_-^f)$. Let $u = \sum_i u_i u_i^f$ be the reduced normal form of u and $d = \max_i ht^f(u_i)$. Using Lemma 7.6 we construct a sequence $y_1, \dots, y_d \in \mathfrak{G}'_+$ such that $\hat{u}_j = \hat{\nu}(y_j \dots y_1 u) = \sum_i [y_j \dots [y_1, u_i] \dots]u_i^f \neq 0$, $[y_j \dots [y_1, u_i] \dots] \in \hat{U}$, $ht^f([y_j \dots [y_1, u_i] \dots]) \leq d - j$, $j = 1, \dots, d$, and for each j, if $\hat{u}_j = \sum_{k\in K_j} \hat{u}_{jk}\hat{u}_{jk}^f$ is the reduced normal form of $\hat{u}_j$, then one can find a pair (k,ℓ) for which $\hat{u}_{jk}^f = \hat{u}_{j-1\ell}^f$. Since $y_d \dots y_1 v = \hat{u}_d v \in N$ and $ht^f(\hat{u}_{dk}) = 0$, we can apply Proposition 7.5 and conclude that $\hat{u}_{dk}^f \hat{v}_\lambda \in N$ for all $k \in K_d$. In particular, $\hat{u}_{d-1\ell}^f \hat{v}_\lambda \in N$ for some $\ell \in K_{d-1}$. Thus $\hat{u}_{d-1}\hat{v}_\lambda -$

$\hat{u}_{d-1\ell}\hat{u}^f_{d-1\ell}\hat{v}_\lambda = \sum_{k\in K_{d-1}\setminus\{\ell\}} \hat{u}_{d-1k}\hat{u}^f_{d-1k}\hat{v}_\lambda \in N$ and $\hat{u}^f_{d-1k}\hat{v}_\lambda \in N$ for all $k \in K_{d-1}$ by induction on $|K_{d-1}|$. We conclude by induction on d that $u^f_i\hat{v}_\lambda \in N$ for all i, which completes the proof of (1). Since $\hat{M}(\lambda) \simeq \hat{U} \otimes_{\mathbb{C}} \tilde{M}^f(\lambda)$ and $L_X(\lambda)$ is the unique irreducible quotient of $\hat{M}(\lambda)$, the statement (2) follows immediately from (1). ◻

Set $\mathfrak{A} = \mathfrak{G}_-(\tilde{\pi}(\lambda))$.

Corollary 7.8. *Let $\lambda \in \mathfrak{H}^*$, $\lambda(c) = 0$ and assume λ is in general position.*

1. *If $N \subset \hat{M}(\lambda)$ then $N \simeq (U(\mathfrak{G})/U(\mathfrak{G})\mathfrak{A}) \otimes_{U(\mathbf{p}_X)} N^f$, where u^+_X acts trivially on N^f.*
2. *$L_X(\lambda) \simeq (U(\mathfrak{G})/U(\mathfrak{G})\mathfrak{A}) \otimes_{U(\mathbf{p}_X)} L^f(\lambda)$, where $\mathbf{u}^+_X$ acts trivially on $L^f(\lambda)$.*

Proof. Since $\mathbf{u}^+_X N^f = \mathbf{u}^+_X L^f(\lambda) = \mathfrak{A}N^f = \mathfrak{A}L^f(\lambda) = 0$, the statements follow from Theorem 7.7. ◻

Denote $\check{M}(\lambda) = M_X(\lambda)/U(\mathfrak{G})\mathfrak{A}v_\lambda$. We have a chain of epimorphisms: $M_X(\lambda) \to \check{M}(\lambda) \to \hat{M}(\lambda) \to L_X(\lambda)$. Thus $L_X(\lambda)$ is the unique irreducible quotient of $\check{M}(\lambda)$ and $\check{M}^f(\lambda) \simeq M^f(\lambda)$.

Corollary 7.9. *Let $\lambda \in \mathfrak{H}^*$, $\lambda(c) = 0$ and λ is in general position. If $N \subset \check{M}(\lambda)$ then $N \simeq (U(\mathfrak{G})/U(\mathfrak{G})\mathfrak{A}) \otimes_{U(\mathbf{p}_X)} N^f$, where $\mathbf{u}^+_X$ acts trivially on N^f.*

Proof. Follows from Corollary 7.8, (1). ◻

Corollary 7.10. *Let μ, $\nu \in \mathfrak{H}^*$, $\mu(c) = \nu(c) = 0$ and assume both μ, ν are in general position.*

1. $\mathrm{Hom}_{\mathfrak{G}}(\hat{M}(\mu), \hat{M}(\nu)) = \mathrm{Hom}_{\tilde{\mathfrak{G}}^f}(\tilde{M}^f(\mu), \tilde{M}^f(\nu))$.
2. $\mathrm{Hom}_{\mathfrak{G}}(\check{M}(\mu), \check{M}(\nu)) \simeq \mathrm{Hom}_{\mathbf{m}}(M^f(\mu), M^f(\nu))$.
3. *Modules $\hat{M}(\mu)$ and $\check{M}(\mu)$ have local composition series and $[\hat{M}(\mu) : L_X(\nu)] = [\tilde{M}^f(\mu) : L^f(\nu)]$, $[\check{M}(\mu) : L_X(\nu)] = [M^f(\mu) : L^f(\nu)]$.*

Proof. Follows from Corollaries 7.8, 7.9 and the fact that both $M^f(\mu)$ and $\tilde{M}^f(\mu)$ have local compositon series [13]. ◻

Corollary 7.11. *Let μ, $\nu \in \mathfrak{H}^*$, $\mu(c) = \nu(c) = 0$ and assume both μ, ν are in general position. Then*

1. $\mathrm{Hom}_{\mathfrak{G}}(\check{M}(\mu), \check{M}(\nu)) \neq 0$ *if and only if* $[\check{M}(\nu) : L_X(\mu)] \neq 0$.
2. $\mathrm{Hom}_{\mathfrak{G}}(\hat{M}(\mu), \hat{M}(\nu)) \neq 0$ *if and only if* $[\hat{M}(\nu) : L_X(\mu)] \neq 0$.

Proof. Since

$$\mathrm{Hom}_{\mathbf{m}}(M^f(\mu), M^f(\nu)) \neq 0 \Leftrightarrow [M^f(\nu) : L^f(\mu)] \neq 0$$

and

$$\mathrm{Hom}_{\tilde{\mathfrak{G}}^f}(\tilde{M}^f(\mu), \tilde{M}^f(\nu)) \neq 0 \Leftrightarrow [\tilde{M}^f(\nu) : L^f(\mu)] \neq 0$$

by [13] (Ch.2.11, Theorem 1), the statements follow from Corollary 7.10. ◻

8. Strong BGG resolution for modules $\hat{M}(\lambda_0)$

In this section we assume that X is connected, i.e. the corresponding Coxeter-Dynkin diagram is connected, and thus $\mathfrak{G}^f$ is the derived algebra of an affine Lie algebra. Let $\bar{\pi}^f$ be a basis of Δ^f containing π^f and $\lambda_0 \in \mathfrak{H}^*$ such that $(\lambda_0, \alpha) = 0$ for all $\alpha \in \bar{\pi}^f$. Let W_X be the Weyl group of $\mathfrak{G}^f$, ℓ be a length function and s_β denote the reflection corresponding to the root β. For w and w' in W_X we write $w \leftarrow w'$ if there exists a root $\beta \in \Delta_+^f \cap \Delta^{re}$ such that $w = s_\beta w'$ and $\ell(w) = \ell(w') + 1$. The Bruhat order on W_X is defined by : $w \leq w'$ if $w = w'$ or if there are $w_1, \dots, w_r \in W_X$ such that $w = w_1 \leftarrow \dots \leftarrow w_r = w'$.

For $w \in W_X$ and $\mu \in \mathfrak{H}^*$, define $w \circ \mu = w(\mu + \rho_X) - \rho_X$, where $\rho_X \in \mathfrak{H}^*$ is any fixed element such that $\rho_X(\alpha) = 1$ for all $\alpha \in \bar{\pi}^f$.

Theorem 8.1 (cf. Theorem 5.2 in [3]). *Let X be connected, $\lambda_0 \in \mathfrak{H}^*, (\lambda_0, \alpha) = 0$ for all $\alpha \in \bar{\pi}^f$ and w, $w' \in W_X$. Then*

$$\dim \operatorname{Hom}_{\mathfrak{G}}(\hat{M}(w' \circ \lambda_0), \hat{M}(w \circ \lambda_0)) = 1 \Leftrightarrow w' \leq w$$
$$\Leftrightarrow [\hat{M}(w \circ \lambda_0) : L_X(w' \circ \lambda_0)] \neq 0.$$

Proof. Since $\operatorname{Hom}_{\mathfrak{G}}(\hat{M}(w' \circ \lambda_0), \hat{M}(w \circ \lambda_0)) \simeq \operatorname{Hom}_{\tilde{\mathfrak{G}}^f}(\tilde{M}^f(w' \circ \lambda_0), \tilde{M}^f(w \circ \lambda_0))$ by Corollary 7.10, (1) and $[\hat{M}(w \circ \lambda_0) : L_X(w' \circ \lambda_0)] = [\tilde{M}^f(w \circ \lambda_0) : L^f(w' \circ \lambda_0)]$ by Corollary 7.10, (3), the statement follows from [14], Theorem 8.15. □

For any $i \in \mathbb{Z}_+$, denote $W_X^{(i)} = \{w \in W_X | \ell(w) = i\}$ and set

$$C_i = \bigoplus_{w \in W_X^{(i)}} \hat{M}(w \circ \lambda_0).$$

If w, $w' \in W_X$ we fix $0 \neq i_{w,w'}(\lambda_0) \in \operatorname{Hom}_{\mathfrak{G}}(\hat{M}(w' \circ \lambda_0), \hat{M}(w \circ \lambda_0))$. Let $d_j : C_j \to C_{j-1}, j \geq 1$, defined by $d_j = \bigoplus b^j_{w,w'} i_{w,w'}(\lambda_0)$, $w \in W_X^{(j)}$, $w' \in W_X^{(j-1)}$, where $b^j_{w,w'} \in \{\pm 1\}$ are defined by [14], Lemma 9.6 if $w \leftarrow w'$ and $b^j_{w,w'} = 0$ otherwise.

Theorem 8.2 (cf. Theorem 5.4 in [3]). *Let X be connected, $\lambda_0 \in \mathfrak{H}^*$, $(\lambda_0, \alpha) = 0$ for all $\alpha \in \bar{\pi}^f$ and $\eta : \hat{M}(\lambda_0) \to L_X(\lambda_0)$ be the canonical projection. Then the sequence*

$$\dots C_j \xrightarrow{d_j} C_{j-1} \to \dots \to C_1 \xrightarrow{d_1} \hat{M}(\lambda_0) \xrightarrow{\eta} L_X(\lambda_0) \to 0, \ (j \geq 1)$$

is exact.

Proof. Follows from Corollary 7.8, (1), Corollary 7.10, (1) and Theorem 9.7 in [14]. □

9. Category $\mathfrak{O}_X(\lambda)$

In this section, following [2], we define certain categories of $\mathfrak{G}$-modules, which contain the Verma type modules and their irreducible quotients, and show that the BGG duality holds in these categories.

Let $\lambda \in \mathfrak{H}^*$, $\lambda(c) = 0$ and λ be in general position. Consider the full subcategory $\mathfrak{O}^f(\lambda)$ of the category of weight **m**-modules, whose objects V satisfy

1. $P(V) \subset \{\mu \in \mathfrak{H}^* \mid \mu \leq \lambda\}$.
2. $\dim V_\mu < \infty$ for all $\mu \in P(V)$.

The category $\mathfrak{O}^f(\lambda)$ is stable under the operations of taking submodules, quotients and finite direct sums. Note that for any $\mu \leq \lambda$, the modules $M^f(\mu)$ and $L^f(\mu)$ are objects of $\mathfrak{O}^f(\lambda)$ and, moreover, the modules $L^f(\mu)$ exhaust all irreducible objects in $\mathfrak{O}^f(\lambda)$.

Recall that $\mathbf{m}$ has a triangular decomposition and thus all the results of [14], sections 4-6, can be applied to the category $\mathfrak{O}^f(\lambda)$. In particular, each module $L^f(\mu)$, $\mu \leq \lambda$, has the indecomposable projective cover $I(\mu)$ by [14], Corollary 4.13, and $I(\mu)$ has a Verma composition series: $I(\mu) = I_0 \supset I_1 \supset \ldots \supset I_\ell \supset 0$, where $I_i/I_{i+1} \simeq M^f(\mu_i)$, $\mu_i \leq \lambda$, $i = 0, \ldots, \ell$, by [14], Corollary 4.10. We denote by $(I(\mu) : M^f(\nu))$ the number of indices i in $\{0, \ldots, \ell\}$ such that $\mu_i = \nu$.

Theorem 9.1 (cf. [14], Theorem 6.4)**.** *Let $\mu \leq \lambda$ and $\nu \leq \lambda$. Then*

$$(I(\mu) : M^f(\nu)) = [M^f(\nu) : L^f(\mu)]$$

The main object of our study in this section is the category $\mathfrak{O}_X(\lambda)$, the full subcategory of the category of weight $\mathfrak{G}$-modules V such that

1. $P(V) \subset \{\mu \in \mathfrak{H}^* \mid \lambda - \mu \in Q_+\}$.
2. $\dim V_\mu < \infty$ for all $\mu \leq \lambda$.
3. The module V is generated by $V^f = \sum_{\mu \leq \lambda} V_\mu$.
4. $\mathfrak{G}_-(\tilde{\pi}(\lambda))v = 0$ for all $v \in V^f$.

Remark 9.2.
1. A similar category with $\lambda(c) \neq 0$ was introduced and studied for non-twisted affine algebras in [2], and in general in [3].
2. The modules $\check{M}(\lambda)$, $\hat{M}(\lambda)$ and $L_X(\lambda)$ are objects of $\mathfrak{O}_X(\lambda)$. Moreover, it follows from Theorem 7.7 that any submodule $N \subset \check{M}(\lambda)$ belongs to $\mathfrak{O}_X(\lambda)$ as well.
3. If λ is not in general position, the category $\mathfrak{O}_X(\lambda)$ may not be closed under the operation of taking of submodules. For example, if $(\lambda, \alpha) = 0$ for all $\alpha \in \pi$ and N is the maximal $\mathbf{m}$-submodule of $M^f(\lambda)$, then $M^f(\lambda)/N \simeq \mathbb{C}$ and any $\mathfrak{G}$-submodule of $\check{M}(\lambda)/U(\mathfrak{G})N$ is not an object of $\mathfrak{O}_X(\lambda)$ [7].
4. If $N \in \mathfrak{O}_X(\lambda)$ then $N^f \in \mathfrak{O}(\lambda)$.

Proposition 9.3. *If V is an irreducible module in $\mathfrak{O}_X(\lambda)$, then $V \simeq L_X(\mu)$ for some $\mu \leq \lambda$.*

Proof. Let V be an irreducible module in $\mathfrak{O}_X(\lambda)$. Then $V^f \in \mathfrak{O}^f(\lambda)$ and V^f is an irreducible $\mathbf{m}$-module. Thus $V^f \simeq L^f(\mu)$ for some $\mu \leq \lambda$. Since $P(V) \subset \{\mu \in \mathfrak{H}^* \mid \lambda - \mu \in Q_+\}$, we conclude that $\mathbf{u}_X^+ v = 0$ for all $v \in V^f$, and hence there exist $\mu \in P(V)$ and a non-zero element $v \in V_\mu^f$ for which $\mathfrak{G}_X^+ v = 0$. This implies that V is a homomorphic image of $\check{M}(\mu)$, and thus $V \simeq L_X(\mu)$. ☐

Proposition 9.4. *The category $\mathfrak{O}_X(\lambda)$ is closed under the operations of taking submodules, quotients and finite direct sums.*

Proof. The proof is based on Theorem 7.7, (1) and is analogous to the proof of Corollary 6.6 in [3]. ☐

We will prove the equivalence of the categories $\mathfrak{O}_X(\lambda)$ and $\mathfrak{O}^f(\lambda)$ for λ in general position. Define an exact functor $F : \mathfrak{O}_X(\lambda) \to \mathfrak{O}^f(\lambda)$ by $F(V) = V^f$ and $F(f) = f|_{V^f}$ for any $V \in \mathfrak{O}_X(\lambda)$ and any $f \in \mathrm{Hom}_{\mathfrak{G}}(V, V')$ in $\mathfrak{O}_X(\lambda)$. Also define an exact functor $Y : \mathfrak{O}^f(\lambda) \to \mathfrak{O}_X(\lambda)$ as follows. Let M and M' be the objects in $\mathfrak{O}^f(\lambda)$ and $g \in \mathrm{Hom}_{\mathbf{m}}(M, M')$. We can make M into a $\mathbf{p}_X$-module with a trivial action of $\mathbf{u}_X^+$

and consider a $\mathfrak{G}$-module $Y(M) = (U(\mathfrak{G})/U(\mathfrak{G})\mathfrak{G}_-(\tilde{\pi}(\lambda))) \otimes_{U(\mathbf{p}_X)} M \in \mathfrak{O}_X(\lambda)$ and $Y(g) = 1 \otimes g$. By Corollary 7.9 we immediately conclude that $Y \circ F(\check{M}(\mu)) \simeq \check{M}(\mu)$ and $F \circ Y(M^f(\mu)) \simeq M^f(\mu)$ for $\mu \leq \lambda$.

Theorem 9.5. *Let $\lambda \in \mathfrak{H}^*$, $\lambda(c) = 0$ and λ in general position. Then the categories $\mathfrak{O}_X(\lambda)$ and $\mathfrak{O}^f(\lambda)$ are equivalent.*

Proof. The proof is absolutely analogous to the proof of Theorem 6.7 in [3]. □

For $\mu \leq \lambda$ denote $I_X(\mu) = Y(I(\mu))$. Then $I_X(\mu)$ is an indecomposable projective cover for $L_X(\mu)$ by Theorem 9.5, and it has a Verma composition series: $I_X(\mu) \supset I_0 \supset I_1 \supset \ldots \supset I_\ell \supset 0$, $I_i/I_{i+1} \simeq \check{M}(\mu_i)$, $\mu_i \leq \lambda$, $i = 0, \ldots, \ell$. Let $(I_X(\mu) : \check{M}(\nu))$ be the number of j's such that $\mu_j = \nu$.

Since any object of $\mathfrak{O}^f(\lambda)$ has a local composition series [13], it follows from Theorem 9.5 that any object $V \in \mathfrak{O}_X(\lambda)$ has a local composition series, and $[V : L_X(\mu)]$ denotes the multiplicity of $L_X(\mu)$ in V.

Theorem 9.6 (BGG duality)**.** *Let $\lambda \in \mathfrak{H}^*$, $\lambda(c) = 0$, λ in general position and $\mu \leq \lambda$, $\nu \leq \lambda$. Then*

$$(I_X(\mu) : \check{M}(\nu)) = [\check{M}(\nu) : L_X(\mu)].$$

Proof. It follows from Theorems 9.5 and 9.1 that $(I_X(\mu) : \check{M}(\nu)) = (I(\mu) : M^f(\nu)) = [M^f(\nu) : L^f(\mu)] = [\check{M}(\nu) : L_X(\mu)]$. □

10. Generalized Verma type modules of level zero

Let $X \neq I \cup \emptyset$, $S \subsetneq \pi^f$, $\lambda \in \mathfrak{H}^*$, $\lambda(c) = 0$ and λ_S is dominant integral. Set $\mathfrak{A} = \mathfrak{G}_-(\tilde{\pi}(\lambda))$ and consider the $\mathfrak{G}$-submodules $M_S = U(\mathfrak{G})(\bar{G}_- \oplus \mathfrak{A})(1 \otimes V_S(\lambda))$ and $M'_S = U(\mathfrak{G})\mathfrak{A}(1 \otimes V_S(\lambda))$ of $M_{X,S}(\lambda)$. Let $\hat{M}_S(\lambda) = M_{X,S}(\lambda)/M_S$ and $\check{M}_S(\lambda) = M_{X,S}(\lambda)/M'_S$. Then both $\hat{M}_S(\lambda)$ and $\check{M}_S(\lambda)$ are weight $\mathfrak{G}$-modules with the unique irreducible quotient $L_X(\lambda)$, and one has the following chain of surjective homomorphisms:

$$\begin{array}{ccccccccc} & & & & \check{M}_S(\lambda) & & & & \\ & & & \nearrow & & \searrow & & & \\ M_X(\lambda) & \to & \check{M}(\lambda) & & & & \hat{M}_S(\lambda) & \to & L_X(\lambda). \\ & & & \searrow & & \nearrow & & & \\ & & & & \hat{M}(\lambda) & & & & \end{array}$$

Clearly, $\check{M}^f_S(\lambda) \simeq M^f_{X,S}(\lambda)$ and $\hat{M}^f_S(\lambda) \simeq M^f_{X,S}(\lambda)/(M^f_{X,S}(\lambda) \cap M_S)$.

Remark 10.1. If $S = \emptyset$ then $\hat{M}_\emptyset(\lambda) \simeq \hat{M}(\lambda)$.

The next statement describes the structure of modules $\hat{M}_S(\lambda)$ and $\check{M}_S(\lambda)$ with λ in general position.

Theorem 10.2. *Let $X \neq I \cup \emptyset$, $S \subsetneq \pi^f$, $\lambda, \mu \in \mathfrak{H}^*$, $\lambda(c) = \mu(c) = 0$, both λ and μ in general position and both λ_S and μ_S dominant integral.*

1. *If $N \subset \hat{M}_S(\lambda)$ (resp. $N \subset \check{M}_S(\lambda)$), then N is generated by N^f and*

$$N \simeq (U(\mathfrak{G}/\mathfrak{U}(\mathfrak{G})\mathfrak{A}) \otimes_{U(\mathbf{p}_X)} N^f,$$

where u^+_X acts trivially on N^f.

2. $\mathrm{Hom}_{\mathfrak{G}}(\check{M}_S(\lambda), \check{M}_S(\mu)) \simeq \mathrm{Hom}_{\mathbf{m}}(M^f_{X,S}(\lambda), M^f_{X,S}(\mu))$,
   $\mathrm{Hom}_{\mathfrak{G}}(\hat{M}_S(\lambda), \hat{M}_S(\mu)) \simeq \mathrm{Hom}_{\tilde{\mathfrak{G}}^f}(\hat{M}^f_S(\lambda), \hat{M}^f_S(\mu))$.
3. *Both* $\check{M}_S(\lambda)$ *and* $\hat{M}_S(\lambda)$ *have local composition series,* $[\check{M}_S(\lambda) : L_X(\mu)] = [M^f_{X,S}(\lambda) : L^f(\mu)]$ *and* $[\hat{M}_S(\lambda) : L_X(\mu)] = [\hat{M}^f_S(\lambda) : L^f(\mu)]$.

Proof. Let $\hat{\tau}_S : \hat{M}(\lambda) \to \hat{M}_S(\lambda)$ (resp. $\check{\tau}_S : \check{M}(\lambda) \to \check{M}_S(\lambda)$) be a natural epimorphism. There exists a submodule $\hat{N} \subset \hat{M}(\lambda)$ (resp. $\check{N} \subset \check{M}(\lambda)$) such that $\hat{\tau}_S(\hat{N}) = N$ (resp. $\check{\tau}_S(\check{N}) = N$) and $\hat{\tau}(\hat{N}^f) = N^f$ (resp. $\check{\tau}_S(\check{N}^f) = N^f$). Thus, statement (1) follows from Corollary 7.8 and Corollary 7.9. Further, (1) implies that $\check{M}_S(\lambda)$ (resp. $\hat{M}_S(\lambda)$) is generated by $M^f_{X,S}(\lambda)$ (resp. $\hat{M}^f_S(\lambda)$), and hence (2) and (3) follow. This completes the proof. □

Remark 10.3. Let $X \neq I \cup \emptyset$, $S \subsetneq \pi^f$, S is connected, $\lambda_0 \in \mathfrak{H}^*$ and $(\lambda_0, \alpha) = 0$ for all $\alpha \in \bar{\pi}^f$. Using [14], Theorem 9.12 and following [3], Theorem 5.7, one can construct the generalized strong BGG resolution for modules $\hat{M}_S(\lambda_0)$. We omit the details.

11. "Truncated" categories $\mathfrak{O}_{X,S}(\lambda, q)$

Let $X \neq I \cup \emptyset$, $S \subsetneq \pi^f$, $\lambda \in \mathfrak{H}^*$, $\lambda(c) = 0$ and $q \in \mathbb{Z}_+$. Define ${}_qQ^+ = \{\mu = \sum m_\alpha \alpha \in Q^f_+ \mid \alpha \in \pi, m_\alpha \in \mathbb{Z}_+, \sum m_\alpha > q\}$, $\Pi = \{\mu \in \mathfrak{H}^* \mid \mu - \lambda = \sum_{\alpha \in \pi} m_\alpha \alpha \in Q^f, m_\alpha \in \mathbb{Z}, (\mu - \lambda)^+ = \sum_{m_\alpha \in \mathbb{Z}_+} m_\alpha \alpha \in Q^f_+ \setminus_q Q^+\}$, $\Pi^S = \{\mu \in \Pi \mid \mu_S$ is integral dominant$\}$. Clearly, there is a one-to-one correspondence between elements of Π^S and irreducible finite-dimensional $\mathbf{m}_S$-modules $V_S(\mu)$ with highest weight $\mu \in \Pi$.

Consider the full subcategory $\mathfrak{O}^f_S = \mathfrak{O}^f_S(\lambda, q)$ of the category of weight $\mathbf{m}$-modules, whose objects V satisfy

1. $\dim V_\mu < \infty$ for all $\mu \in \mathfrak{H}^*$.
2. V is a direct sum of $V_S(\mu)'s$, $\mu \in \Pi^S$ (cf. [14] and [3]).

The category $\mathfrak{O}^f_S$ is stable under the operations of taking submodules, quotients and finite direct sums. Note that $M^f_{X,S}(\mu)$ and $L^f(\mu)$ belong to $\mathfrak{O}^f_S$ for all $\mu \in \Pi^S$, and the modules $L^f(\mu)$, $\mu \in \Pi^S$, exhaust all irreducible objects in $\mathfrak{O}^f_S$.

Fix $\mu \in \Pi^S$ and set $D_{X,S} = \mathbf{u}^+_{X,S} \oplus \bar{G}_+$, $U(D_{X,S})^{(\mu)} = \sum U(D_{X,S})_\alpha, \alpha \in Q$, $\nu + \alpha \notin \Pi$ for all $\nu \in P(V_S(\mu))$. Then $\mathfrak{A}(\mu, S) = (U(D_{X,S})/U(D_{X,S})^{(\mu)}) \otimes V_S(\mu)$ is a weight $\mathbf{p}_{X,S} \oplus \bar{G}_+$-module where $D_{X,S}$ acts on the left and $\mathbf{m}_S$ by the tensor product action. Let $\bar{A}(\mu, S) = \sum A(\mu, S)_\nu$, $\nu \in \mathfrak{H}^* \setminus \Pi$, and

$$\tilde{\mathfrak{A}}(\mu, S) = \mathfrak{A}(\mu, S)/U(\mathbf{p}_{X,S} \oplus \bar{G}_+)\bar{\mathfrak{A}}(\mu, S).$$

Then $P^S(\mu) = U(\mathbf{m}) \otimes_{U(\mathbf{p}_{X,S} \oplus \bar{G}_+)} \tilde{\mathfrak{A}}(\mu, S)$ is a projective object in $\mathfrak{O}^f_S$. Again, by [14], Corollary 4.13, there is a one-to-one correspondence between the irreducible objects in $\mathfrak{O}^f_S$ and the indecomposable direct summands of the $P^S(\mu)$'s, $\mu \in \Pi^S$. Let $I^S(\mu)$ be the indecomposable projective cover for $L^f(\mu)$.

By [14], Corollary 4.10, the module $I^S(\mu)$ has a generalized Verma composition series, i.e. there exists a filtration $I^S(\mu) = I_0 \supset I_1 \supset \ldots \supset I_\ell \supset 0$ where $I_i/I_{i+1} \simeq M^f_{X,S}(\mu_i)$, $\mu_i \in \Pi^S$. Denote by $(I^S(\mu) : M^f_{X,S}(\nu))$ the number of i's such that $\nu = \mu_i$.

Theorem 11.1 (cf. [14], Theorem 6.4)**.** *If $\mu, \nu \in \Pi^S$, then* $(I^S(\mu) : M^f_{X,S}(\nu)) = [M^f_{X,S}(\nu) : L^f(\mu)]$.

Now suppose that λ is in general position, and consider the full subcategory $\mathfrak{O}_{X,S} = \mathfrak{O}_{X,S}(\lambda, q)$ of the category of weight $\mathfrak{G}$-modules V such that

1. $P(V) \subset \cup_{\mu \in \Pi} \{\nu \in \mathfrak{H}^* \mid \mu - \nu \in Q_+\}$.
2. $\dim V_\mu < \infty$ for all $\mu \in \Pi$.
3. The module V is generated by $V^f = \sum_{\mu \in \Pi} V_\mu$.
4. V^f is a direct sum of $V_S(\mu)$'s, $\mu \in \Pi^S$.
5. $\mathfrak{G}_-(\tilde{\pi}(\lambda))v = 0$ for all $v \in V^f$.

Remark 11.2. 1. If $\mu \in \Pi^S$, then the modules $\check{M}_S(\mu)$, $\hat{M}_S(\mu)$ and $L_X(\mu)$ are objects of the category $\mathfrak{O}_{X,S}$.
2. If $V \in \mathfrak{O}_{X,S}$ then $V^f \in \mathfrak{O}^f_S$.
3. If $V \in \mathfrak{O}_{X,S}$ is irreducible then $V \simeq L_X(\mu)$ for some $\mu \in \Pi^S$. The proof is analogous to the proof of Proposition 9.3.
4. The category $\mathfrak{O}_{X,S}$ is closed under the operations of taking submodules, quotients and finite direct sums (cf. Proposition 3.4).
5. $\mathfrak{O}_{X,\emptyset}(\lambda, 0) = \mathfrak{O}_X(\lambda)$.

Theorem 11.3. *Let $\lambda \in \mathfrak{H}^*$, $\lambda(c) = 0$, λ in general position and $q \in \mathbb{Z}_+$. Then the categories $\mathfrak{O}_{X,S}(\lambda, q)$ and $\mathfrak{O}^f_S(\lambda, q)$ are equivalent.*

Proof. As in section 9 define exact functors $F : \mathfrak{O}_{X,S} \to \mathfrak{O}^f_S$ and $Y : \mathfrak{O}^f_S \to \mathfrak{O}_{X,S}$. Then $Y \circ F(\check{M}_S(\mu)) \simeq \check{M}_S(\mu)$ and $F \circ Y(M^f_{X,S}(\mu)) \simeq M^f_{X,S}(\mu)$ for all $\mu \in \Pi^S$ by Theorem 10.2. Moreover, F and Y establish an equivalence of $\mathfrak{O}_{X,S}$ and $\mathfrak{O}^f_S$. The proof follows the proof of Theorem 6.7 in [3]. □

For $\mu \in \Pi^S$ let $I^S_X(\mu) = Y(I^S(\mu))$. Then $I^S_X(\mu)$ is an indecomposable projective cover for $L_X(\mu)$ in $\mathfrak{O}_{X,S}$ by Theorem 11.3, and it has a generalized Verma composition series with factors $\check{M}_S(\mu_i)$, $\mu_i \in \Pi^S$. Denote by $[I^S_X(\mu) : \check{M}_S(\nu)]$ the multiplicity of $\check{M}_S(\nu)$ in a generalized Verma composition series for $I^S_X(\mu)$ and by $(\check{M}_S(\nu) : L_X(\mu))$ the multiplicity of $L_X(\mu)$ in a local composition series for $\check{M}_S(\nu)$.

Theorem 11.4 (BGG duality)**.** *If λ is in general position and $\mu, \nu \in \Pi^S$, then*

$$[I^S_X(\mu) : \check{M}_S(\nu)] = (\check{M}_S(\nu) : L_X(\mu)).$$

Proof. Follows from Theorems 11.3 and 11.1. □

12. Some subcategories of $\mathfrak{O}_{X,S}(\lambda, q)$

Consider the full subcategory $\bar{\mathfrak{O}}^f_S = \bar{\mathfrak{O}}^f_S(\lambda, q) \subset \mathfrak{O}^f_S(\lambda, q)$ consisting of **m**-modules V such that $\bar{G}v = 0$ for all $v \in V$, and the full subcategory $\bar{\mathfrak{O}}_{X,S} = \bar{\mathfrak{O}}_{X,S}(\lambda, q) \subset \mathfrak{O}_{X,S}(\lambda, q)$ consisting of $\mathfrak{G}$-modules M such that $M^f \in \bar{\mathfrak{O}}^f_S$. Obviously, $\hat{M}_S(\mu)$ and $L_X(\mu)$ are objects of $\bar{\mathfrak{O}}_{X,S}$ for any $\mu \in \Pi^S$.

Let $\mu \in \Pi^S$. If we replace $\bar{G}_+$ by $\bar{G}$ in the construction of $P^S(\mu)$ we obtain a projective module $\bar{P}^S(\mu)$ in $\bar{\mathfrak{O}}^f_S$, whose indecomposable summands exhaust all indecomposable projectives in $\bar{\mathfrak{O}}^f_S$. Let $\bar{I}^S(\mu)$ be the indecomposable projective cover for $L^f(\mu)$.

Since $F(M) \in \bar{\mathfrak{O}}^f_S$ for any $M \in \bar{\mathfrak{O}}_{X,S}$ and $Y(N) \in \bar{O}_{X,S}$ for any $N \in \bar{\mathfrak{O}}^f_S$, the functors F and Y induce the exact functors $\bar{F} : \bar{\mathfrak{O}}_{X,S} \to \bar{\mathfrak{O}}^f_S$ and $\bar{Y} : \bar{\mathfrak{O}}^f_S \to \bar{\mathfrak{O}}_{X,S}$. Denote $\bar{I}^S_X(\mu) = Y(I^S(\mu))$, the indecomposable projective cover for $L_X(\mu)$ in $\bar{\mathfrak{O}}_{X,S}$.

Theorem 12.1. *Let $\mu, \nu \in \Pi^S(\lambda, q)$.*

1. $[\bar{I}^S(\mu) : \hat{M}^f_S(\nu)] = (\hat{M}^f_S(\nu) : L^f(\mu))$.
2. *If λ is in general position, then the categories $\bar{\mathfrak{O}}_{X,S}$ and $\bar{\mathfrak{O}}^f_S$ are equivalent.*
3. *If λ is in general position, then*

$$[\bar{I}^S_X(\mu) : \hat{M}_S(\nu)] = (\hat{M}_S(\nu) : L_X(\mu)).$$

Proof. The proof of (1) follows the general lines of [14], Theorem 6.4; the proof of (2) is analogous to the proof of Theorem 6.7 in [3]; and (3) follows from (1) and (2). □

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